2018+ Putnam Problems

Below are a few problems from the Putnam exams of 2018–2021. These are 2nd or 3rd easiest, in my opinion. This is "surprise" format: we'll look at these without prior announcement. For solutions see the Putnam Archive.

Problems:

2018 A2: Let $S_1, S_2, \ldots, S_{2^n-1}$ be the nonempty subsets of $\{1,2,\ldots,n\}$ in some order, and let M be the $(2^n-1)\times(2^n-1)$ matrix whose (i,j) entry is

$$m_{ij} = \begin{cases} 0 & \text{if } S_i \cap S_j = \emptyset; \\ 1 & \text{otherwise.} \end{cases}$$

Calculate the determinant of M.

2018 B3: Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides 2^n-1, and $n-2$ divides 2^n-2.

2019 A3: Given real numbers $b_0, b_1, \ldots, b_{2019}$ with $b_{2019} \neq 0$, let $z_1, z_2, \ldots, z_{2019}$ be the roots in the complex plane of the polynomial

$$P(z) = \sum_{k=0}^{2019} b_k z^k.$$

Let $\mu = (|z_1| + \cdots + |z_{2019}|)/2019$ be the average of the distances from $z_1, z_2, \ldots, z_{2019}$ to the origin. Determine the largest constant M such that $\mu \geq M$ for all choices of $b_0, b_1, \ldots, b_{2019}$ that satisfy

$$1 \leq b_0 < b_1 < b_2 < \cdots < b_{2019} \leq 2019.$$

2019 B2: For all $n \geq 1$, let

$$a_n = \sum_{k=1}^{n-1} \frac{\sin\left(\frac{(2k-1)\pi}{2n}\right)}{\cos^2\left(\frac{(k-1)\pi}{2n}\right)\cos^2\left(\frac{k\pi}{2n}\right)}.$$

Determine

$$\lim_{n\to\infty} \frac{a_n}{n^3}.$$

2020 A2: Let k be a nonnegative integer. Evaluate

$$\sum_{j=0}^{k} 2^{k-j}\binom{k+j}{j}.$$

2020 B2: Let k and n be integers with $1 \leq k < n$. Alice and Bob play a game with k pegs in a line of n holes. At the beginning of the game, the pegs occupy the k leftmost holes. A legal move consists of moving a single peg to any vacant hole that is further to the right. The players alternate moves, with Alice playing first. The game ends when the pegs are in the k rightmost holes, so whoever is next to play cannot move and therefore loses. For what values of n and k does Alice have a winning strategy?

2021 A2: For every positive real number x, let

$$g(x) = \lim_{r\to 0}((x+1)^{r+1} - x^{r+1})^{\frac{1}{r}}.$$

Find $\lim_{x\to\infty} \frac{g(x)}{x}$.

2021 B2: Determine the maximum value of the sum

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n}(a_1 a_2 \cdots a_n)^{1/n}$$

over all sequences $a_1, a_2, a_3, \cdots$ of nonnegative real numbers satisfying

$$\sum_{k=1}^{\infty} a_k = 1.$$