

200[0-3] Putnam Problems

Below are a few problems selected from the Putnam exams of 2000 to 2003. I avoided the “easiest” problems, which are more widely discussed. These are 2nd or 3rd easiest, in my opinion. This is “surprise” format: we’ll look at these without prior announcement. For solutions see the [Putnam Archive](#).

Problems:

2000 A3: The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5, and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

2000 B2: Prove that the expression

$$\frac{\gcd(m,n)}{n} \binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

2001 A2: You have coins $C_1, C_2, \dots, C_n$. For each k , C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n .

2001 B3: For any positive integer n , let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

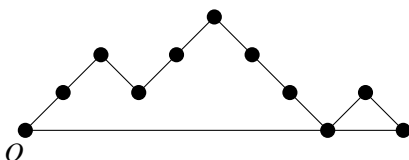
2002 A4: In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1’s and four 0’s. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

2002 B2: Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

Show that the player who signs first will always win by playing as well as possible.

2003 A5: A Dyck n -path is a lattice path of n upsteps $(1, 1)$ and n downsteps $(1, -1)$ that starts at the origin O and never dips below the x -axis. A return is a maximal sequence of contiguous downsteps that terminates on the x -axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.



Show that there is a one-to-one correspondence between the Dyck n -paths with no return of even length and the Dyck $(n-1)$ -paths.

2003 B2: Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}$, form a new sequence of $n-1$ entries $\frac{3}{4}, \frac{5}{12}, \dots, \frac{2n-1}{2n(n-1)}$ by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n-2$ entries, and continue until the final sequence produced consists of a single number x_n . Show that $x_n < 2/n$.