ICMT Division A Sample Problems

July 10, 2025

1. Evaluate
$$\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+4)!}.$$

2. Compute the integral
$$\int_0^1 \int_0^1 \frac{1}{1-xy}\, dx\, dy.$$

3. A group G has four distinct generators a, b, c, d (so that every element of G is a product of these four elements or their inverses). The group G also has the 16 relations $xyxy = 1_G$ for all $x, y \in \{a, b, c, d\}$, where 1_G is the group identity. What is the minimum number of relations that need to be added to G to make the number of distinct elements of G finite?

4. What is the expected number of positive divisors of an integer chosen uniformly at random from 1 to 1000000, inclusive? If A is your answer and C is the correct value, your score on this question will be $\left\lfloor 25 \cdot e^{-38|A-C|^2} \right\rfloor$.

5. Compute the value of
$$\int_{-\infty}^{\infty} \frac{e^{2026iz}}{z^4 + 2z^2 + 1}\, dz.$$

6. Let $\mathcal{M} = \mathbb{F}_4^{4\times 4}$ be the set of 4×4 matrices with elements in $\mathbb{F}_4$. Let $\mathcal{N}$ be the nilpotent subgroup of $\mathcal{M}$; i.e.
$$\mathcal{N} = \{M \in \mathcal{M} : \exists n \in \mathbb{N},\, M^n = 0\}.$$
Define the equivalence relation $A \sim B$ for matrices in $\mathcal{M}$ if there exists invertible $V \in \mathcal{M}$ such that $AV = VB$. Compute $|\mathcal{N}/\sim|$.

7. Define S_N to be the number of integral solutions (x, y, z) to the equation $x + y + z = 0$ such that $-N \le x, y, z \le N$ and $\gcd(x, y) = \gcd(y, z) = \gcd(z, x) = 1$. Compute
$$\lim_{N\to\infty} \frac{S_N}{N^2}.$$

8. For $t \ge 0$, the polynomial equation
$$x^5 - tx - 1 = 0$$
has one positive real solution; denote this solution by $\lambda(t)$. Estimate
$$\int_0^{1/25} \frac{\lambda(5t)^2}{\lambda(5t)^4 - t}\, dt.$$
If A is your answer and C is the correct value, your score on this question will be $\left\lfloor 25 \cdot e^{-\left(\frac{A-C}{0.0035}\right)^2} \right\rfloor$.

9. Let S be the set of all differentiable functions with continuous derivatives $f : [0, 1] \to \mathbb{R}$ such that
$$\int_0^1 f(x)\, dx = \int_0^1 x f(x)\, dx = 1.$$
Compute
$$\inf_{f\in S} \left(\int_0^1 (f'(x))^2\, dx \right).$$

10. Compute the number of ordered pairs (g, h) of elements of S_5, the group of permutations of 5 elements, such that $gh = hg$.