Calculus Surprise Putnam Problems

Below are some problems from recent Putnam exams. The first three are leftover problems from our combinatorics session. After that are several newly selected calculus problems.

This is "surprise" format: we'll look at them without prior announcement. See if you can make progress on any of them. For solutions, see the Putnam Archive.

Leftover Combinatorial Problems:

2013 B3. Let $\mathscr{P}$ be a nonempty collection of subsets of $\{1,\dots,n\}$ such that:

(i) if $S,S' \in \mathscr{P}$, then $S\cup S' \in \mathscr{P}$ and $S\cap S' \in \mathscr{P}$, and
(ii) if $S \in \mathscr{P}$ and $S \neq \emptyset$, then there is a subset $T \subset S$ such that $T \in \mathscr{P}$ and $|T| = |S|-1$.

Suppose $f : \mathscr{P} \to \mathbb{R}$ is a function such that $f(\emptyset) = 0$ and $f(S\cup S') = f(S)+f(S')-f(S\cap S')$ for all $S,S' \in \mathscr{P}$. Must there exist real numbers $f_1,\dots,f_n$ such that $f(S) = \sum_{i\in S} f_i$ for every $S \in \mathscr{P}$?

2017 A4. A class with $2N$ students took a quiz, on which the possible scores were $0,1,\dots,10$. Each of these scores occurred at least once, and the average score was exactly 7.4. Show that the class can be divided into two groups of N students in such a way that the average score for each group was exactly 7.4.

2022 B3. Assign to each positive real number a color, either red or blue. Let D be the set of all distances $d > 0$ such that there are two points of the same color at distance d apart. Recolor the positive reals so that the numbers in D are red and the numbers not in D are blue. If we iterate this recoloring process, will we always end up with all the numbers red after a finite number of steps?

Newly Selected Calculus Problems:

2010 A2. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n)-f(x)}{n}$$

for all real numbers x and all positive integers n.

2011 B3. Let f and g be (real-valued) functions defined on an open interval containing 0, with g nonzero and continuous at 0. If fg and f/g are differentiable at 0, must f be differentiable at 0?

2015 B1. Let f be a three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that f has at least five distinct real zeros. Prove that $f+6f'+12f''+8f'''$ has at least two distinct real zeros.

2017 A3. Let a and b be real numbers with $a < b$, and let f and g be continuous functions from $[a,b]$ to $(0,\infty)$ such that $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$ but $f \neq g$. For every positive integer n, define

$$I_n = \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n}\,dx.$$

Show that $I_1, I_2, I_3, \dots$ is an increasing sequence with $\lim_{n\to\infty} I_n = \infty$.

2019 A4. Let f be a continuous real-valued function on $\mathbb{R}^3$. Suppose that for every sphere S of radius 1, the integral of $f(x,y,z)$ over the surface of S equals 0. Must $f(x,y,z)$ be identically 0?