Combinatorial Surprise Putnam Problems

Below are a few combinatorial problems from recent Putnam exams. These were not the easiest problems in their sessions; rather they were 2nd or 3rd easiest.

This is “surprise” format: we’ll look at them without prior announcement. See if you can make progress on any of them. These problems and solutions are from the Putnam Archive.

Problems:

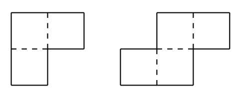
2013 A–2. Let S be the set of all positive integers that are *not* perfect squares. For n in S, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minumum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3$, $2 \cdot 4$, $2 \cdot 5$, $2 \cdot 3 \cdot 4$, $2 \cdot 3 \cdot 5$, $2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function f from S to the integers is one-to-one.

2013 B–3. Let $\mathscr{P}$ be a nonempty collection of subsets of $\{1, \ldots, n\}$ such that:

(i) if $S, S' \in \mathscr{P}$, then $S \cup S' \in \mathscr{P}$ and $S \cap S' \in \mathscr{P}$, and
(ii) if $S \in \mathscr{P}$ and $S \neq \emptyset$, then there is a subset $T \subset S$ such that $T \in \mathscr{P}$ and $|T| = |S| - 1$.

Suppose $f : \mathscr{P} \to \mathbb{R}$ is a function such that $f(\emptyset) = 0$ and $f(S \cup S') = f(S) + f(S') - f(S \cap S')$ for all $S, S' \in \mathscr{P}$. Must there exist real numbers $f_1, \ldots, f_n$ such that $f(S) = \sum_{i \in S} f_i$ for every $S \in \mathscr{P}$?

2016 A–4. Consider a $(2m-1) \times (2n-1)$ rectangular region, where m and n are integers such that $m, n \geq 4$. This region is to be tiled using tiles of the two types shown:



(The dotted lines divide the tiles into 1×1 squares.) The tiles may be rotated and reflected, as long as their sides are parallel to the sides of the rectangular region. They must all fit within the region, and they must cover it completely without overlapping.

What is the minimum number of tiles required to tile the region?

2017 A–4. A class with $2N$ students took a quiz, on which the possible scores were $0, 1, \ldots, 10$. Each of these scores occurred at least once, and the average score was exactly 7.4. Show that the class can be divided into two groups of N students in such a way that the average score for each group was exactly 7.4.

2022 B–3. Assign to each positive real number a color, either red or blue. Let D be the set of all distances $d > 0$ such that there are two points of the same color at distance d apart. Recolor the positive reals so that the numbers in D are red and the numbers not in D are blue. If we iterate this recoloring process, will we always end up with all the numbers red after a finite number of steps?