

# Putnam Practice Session

## Generating Functions: Tools, Facts, and Applications

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### Overview & Pedagogical Goals

Generating functions are among the most powerful algebraic unified frameworks in combinatorics, probability, and number theory. By encoding a sequence  $(a_0, a_1, a_2, \dots)$  into the formal coefficients of a power series  $A(x) = \sum_{n=0}^{\infty} a_n x^n$ , combinatorial operations (such as convolution, subset selection, and recurrence relation solving) translate directly into algebraic manipulations.

This packet is organized into three parts:

1. **Study Guide & Fundamentals:** Key series expansions, operational properties, bivariate series, and introductory exercises with solutions.
  2. **Selected Putnam Problems:** Five accessible Putnam problems (ranging from 2003 to 2024) whose solutions leverage ordinary and exponential generating functions.
  3. **Solutions & Mathematical Commentary:** Rigorous proofs and step-by-step commentary for all five selected problems.
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## 1 Study Guide & Core Principles

### 1.1 1. Ordinary and Exponential Generating Functions

**Definition 1** (Ordinary Generating Function (OGF)). *The **ordinary generating function** of a sequence  $(a_0, a_1, a_2, \dots)$  is the formal power series*

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

*The operator  $[x^n]A(x)$  denotes the coefficient of  $x^n$  in  $A(x)$ , so  $[x^n]A(x) = a_n$ .*

**Definition 2** (Exponential Generating Function (EGF)). *The **exponential generating function** of a sequence  $(a_0, a_1, a_2, \dots)$  is defined as*

$$E(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

*EGFs are especially well-suited for structures involving labeled objects, permutations, and set partitions.*

## 1.2 2. Fundamental Expansions and Operational Rules

**Theorem 1** (Essential Power Series Expansions). *The following formal series expansions are foundational in competition problem-solving:*

1. **Geometric Series:**

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (|x| < 1).$$

2. **Negative Binomial Expansion:** For any integer  $k \geq 1$ ,

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n.$$

3. **Generalized Binomial Theorem:** For any real  $\alpha \in \mathbb{R}$ ,

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad \text{where } \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

4. **Exponential and Logarithmic Series:**

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \ln\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

**Theorem 2** (Cauchy Convolution Product). *If  $A(x) = \sum_{n=0}^{\infty} a_n x^n$  and  $B(x) = \sum_{n=0}^{\infty} b_n x^n$ , then their product is*

$$A(x)B(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{where } c_n = \sum_{k=0}^n a_k b_{n-k}.$$

*In particular, multiplying  $A(x)$  by  $\frac{1}{1-x}$  produces the sequence of partial sums:*

$$\frac{A(x)}{1-x} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k \right) x^n.$$

## 1.3 3. Simple Introductory Exercises with Solutions

**Example 1** (Solving Linear Recurrences via OGF). *Find a closed form for the  $n$ -th term of the Fibonacci sequence  $F_0 = 0, F_1 = 1$ , and  $F_{n+1} = F_n + F_{n-1}$  for  $n \geq 1$ .*

*Solution.* Let  $F(x) = \sum_{n=0}^{\infty} F_n x^n$ . Multiply the recurrence relation  $F_n = F_{n-1} + F_{n-2}$  by  $x^n$  and sum over  $n \geq 2$ :

$$\sum_{n=2}^{\infty} F_n x^n = x \sum_{n=2}^{\infty} F_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} F_{n-2} x^{n-2}.$$

This translates algebraically to

$$F(x) - x = xF(x) + x^2 F(x) \implies F(x)(1-x-x^2) = x \implies F(x) = \frac{x}{1-x-x^2}.$$

Factoring  $1-x-x^2 = (1-\alpha x)(1-\beta x)$  where  $\alpha = \frac{1+\sqrt{5}}{2}$  and  $\beta = \frac{1-\sqrt{5}}{2}$ , partial fraction decomposition yields

$$F(x) = \frac{1}{\sqrt{5}} \left( \frac{1}{1-\alpha x} - \frac{1}{1-\beta x} \right) = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} (\alpha^n - \beta^n) x^n.$$

Extracting the coefficient of  $x^n$  gives Binet's formula:  $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$ . □

**Example 2** (Partition into Coin Types). *Find the ordinary generating function for the number of ways  $a_n$  to make  $n$  cents change using 1-cent and 2-cent coins.*

*Solution.* The choices for 1-cent coins are encoded by  $1 + x + x^2 + \cdots = \frac{1}{1-x}$ , and the choices for 2-cent coins are encoded by  $1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}$ . By independence,

$$A(x) = \frac{1}{(1-x)(1-x^2)} = \frac{1}{(1-x)^2(1+x)}.$$

Using partial fraction decomposition:

$$A(x) = \frac{1/4}{1-x} + \frac{1/2}{(1-x)^2} + \frac{1/4}{1+x}.$$

Extracting the coefficient  $[x^n]A(x)$  yields:

$$a_n = \frac{1}{4} + \frac{1}{2}(n+1) + \frac{1}{4}(-1)^n = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

□

## 2 Selected Putnam Problems

Here are five accessible Putnam problems whose solutions rely cleanly on ordinary or exponential generating functions.

### Problem 1 (Putnam 2020, Problem A2)

Let  $k$  be a nonnegative integer. Evaluate

$$\sum_{j=0}^k 2^{k-j} \binom{k+j}{j}.$$

### Problem 2 (Putnam 2005, Problem B4)

For positive integers  $m$  and  $n$ , let  $f(m, n)$  denote the number of  $n$ -tuples  $(x_1, x_2, \dots, x_n)$  of integers such that  $|x_1| + |x_2| + \dots + |x_n| \leq m$ . Show that  $f(m, n) = f(n, m)$ .

### Problem 3 (Putnam 2003, Problem A6)

For a set  $S$  of nonnegative integers, let  $r_S(n)$  denote the number of ordered pairs  $(s_1, s_2)$  such that  $s_1 \in S, s_2 \in S, s_1 \neq s_2$ , and  $s_1 + s_2 = n$ . Is it possible to partition the nonnegative integers into two sets  $A$  and  $B$  in such a way that  $r_A(n) = r_B(n)$  for all  $n$ ?

## SHORT VERSION ENDS HERE!

The “short” version of this document ends here, I will distribute it by email. The “full” version includes two more problems and full solutions. I’ll release it online during our meeting.