Solutions to the 85th William Lowell Putnam Mathematical Competition
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A1 The answer is $n=1$. When $n=1$, $(a,b,c)=(1,2,2)$ is a solution to the given equation. We claim that there are no solutions when $n \geq 2$.

For $n=2$, suppose that we have a solution to $2a^2+3b^2=4c^2$ with $a,b,c \in \mathbb{N}$. By dividing each of a,b,c by $\gcd(a,b,c)$, we obtain another solution; thus we can assume that $\gcd(a,b,c)=1$. Note that we have $a^2+c^2 \equiv 0 \pmod{3}$, and that only 0 and 1 are perfect squares mod 3; thus we must have $a^2 \equiv c^2 \equiv 0 \pmod{3}$. But then a,c are both multiples of 3; it follows from $b^2 = 12(c/3)^2 - 6(a/3)^2$ that b is a multiple of 3 as well, contradicting our assumption that $\gcd(a,b,c)=1$.

For $n \geq 3$, suppose that $2a^n+3b^n=4c^n$. As in the previous case, we can assume $\gcd(a,b,c)=1$. Since $3b^n = 4c^n - 2a^n$, b must be even. We can then write $a^n + 12(b/2)^n = 2c^n$, and so a must be even. Then $4(a/2)^n + 6(b/2)^n = c^n$, and c must be even as well. This contradicts our assumption that $\gcd(a,b,c)=1$.

A2 The answer is $p(x) = \pm x + c$ for any $c \in \mathbb{R}$. Note that any such polynomial works: if $p(x)=x+c$ then $p(x)-x=c$, while if $p(x)=-x+c$ then $p(p(x))-x=0$.

We will show that these are the only polynomials $p(x)$ such that $p(p(x))-x$ is divisible by $(r(x))^2$, where $r(x)=p(x)-x$. Suppose that $p(x)$ satisfies this condition. We first claim that $r(x)$ has only simple roots (roots of multiplicity 1) in $\mathbb{C}$. Indeed, suppose α is a multiple root of $r(x)$ of order d for some $d \geq 2$, and write $r(x) = (x-\alpha)^d s(x)$ for some polynomial $s(x)$ with $s(\alpha) \neq 0$. Then applying the formula $p(x) = (x-\alpha)^d s(x) + x$ twice and collecting terms yields

$$p(p(x)) - x = (x-\alpha)^d \cdot \left[((x-\alpha)^{d-1}s(x)+1)^d s((x-\alpha)^d s(x)+x) + s(x)\right].$$

Now setting $x=\alpha$ in the expression in square brackets gives $2s(\alpha) \neq 0$. It follows that $(x-\alpha)^{d+1}$ does not divide $p(p(x))-x$, whereas it does divide $(r(x))^2 = (x-\alpha)^{2d}(s(x))^2$, contradiction. This proves the claim.

Now suppose that α is a root of $r(x)$; then $(x-\alpha)^2$ divides $(r(x))^2$ and thus divides $p(p(x))-x$ as well. It follows that $x-\alpha$ divides the derivative of $p(p(x))-x$, which is $p'(x)p'(p(x))-1$, whence $0 = p'(\alpha)p'(p(\alpha))-1 = (p'(\alpha))^2 - 1 = r'(\alpha)(r'(\alpha)+2)$. But $r'(\alpha) \neq 0$ since every root of $r(x)$ is simple, and thus $r'(\alpha)+2=0$. We conclude that every root of $r(x)$ is a root of $r'(x)+2$; again since every root of $r(x)$ is simple, it follows that $r(x)$ divides $r'(x)+2$.

If $r(x)$ is constant, then clearly $p(x)=x+c$ for some constant c. If $r(x)$ is nonconstant, then the degree of $r(x)$ is greater than the degree of $r'(x)+2$; thus $r(x)$ can only divide $r'(x)+2$ if $r'(x)+2=0$, in which case $p(x)=-x+c$ for some constant c. This completes the proof.

A3 Yes, such a,b,c,d exist: we take

$$(a,b)=(2,1), \qquad (c,d)=(1,2).$$

We will represent T as an $3 \times n$ array (3 rows, n columns) of integers in which each of $1,\dots,3n$ occurs exactly once and the rows and columns are strictly increasing; we will specialize to $n=2024$ at the end.

We first note that $T(1,1)=1$ and $2 \in \{T(1,2),T(2,1)\}$. From this, it follows that $T(2,1)<T(1,2)$ if and only if $T(2,1)=2$.

We next recall a restricted form of the *hook length formula* (see the first remark for a short proof of this restricted version and the second remark for the statement of the general formula). Consider more generally an array consisting of (up to) three rows of lengths $n_1 \geq n_2 \geq n_3 \geq 0$, aligned at the left. Let $f(n_1,n_2,n_3)$ be the number of ways to fill this array with a permutation of the numbers $1,\dots,n_1+n_2+n_3$ in such a way that each row increases from left to right and each column increases from top to bottom. The hook length formula then shows that $f(n_1,n_2,n_3)$ equals

$$\frac{(n_1-n_2+1)(n_1-n_3+2)(n_2-n_3+1)(n_1+n_2+n_3)!}{(n_1+2)!(n_2+1)!n_3!}.$$

We then note that if $T(2,1)=2$, we obtain a array with row lengths $n,n-1,n-1$ by removing 1 and 2, relabeling each remaining i as $3n+1-i$, and reflecting in both axes. The probability that $T(2,1)<T(1,2)$ is thus

$$\begin{aligned}\frac{f(n,n-1,n-1)}{f(n,n,n)} &= \frac{(2)(3)(n+1)n}{(1)(2)(3n)(3n-1)} \\ &= \frac{n+1}{3n-1} = \frac{1}{3} + \frac{4}{9n-3};\end{aligned}$$

this is always greater than $\frac{1}{3}$, and for $n=2024$ it is visibly less than $\frac{2}{3}$.

Remark. We prove the claimed formula for $f(n_1,n_2,n_3)$ by induction on $n_1+n_2+n_3$. To begin with, if $n_2=n_3=0$, then the desired count is indeed $f(n_1,0,0)=1$. Next, suppose $n_2>0, n_3=0$. The entry n_1+n_2 must go at the end of either the first or second row; counting ways to complete the diagram from these starting points yields

$$f(n_1,n_2,0) = f(n_1-1,n_2,0) + f(n_1,n_2-1,0).$$

(This works even if $n_1 = n_2$, in which case the first row is not an option but correspondingly $f(n_2-1,n_2,0) = 0$.) The induction step then follows from the identity

$$\frac{(n_1-n_2)(n_1+1)+(n_1-n_2+2)n_2}{(n_1-n_2+1)(n_1+n_2)} = 1.$$

(As an aside, the case $n_1 = n_2, n_3 = 0$ recovers a standard interpretation of the Catalan numbers.)

Finally, suppose $n_3 > 0$. We then have

$$\begin{aligned} &f(n_1,n_2,n_3) \\ &= f(n_1-1,n_2,n_3)+f(n_1,n_2-1,n_3)+f(n_1,n_2,n_3-1), \end{aligned}$$

and the induction step now reduces to the algebraic identity

$$\begin{aligned} &\frac{(n_1-n_2)(n_1-n_3+1)(n_1+2)}{(n_1-n_2+1)(n_1-n_3+2)(n_1+n_2+n_3)} \\ &+ \frac{(n_1-n_2+2)(n_2-n_3)(n_2+1)}{(n_1-n_2+1)(n_2-n_3+1)(n_1+n_2+n_3)} \\ &+ \frac{(n_1-n_3+3)(n_2-n_3+2)n_3}{(n_1-n_3+2)(n_2-n_3+1)(n_1+n_2+n_3)} = 1. \end{aligned}$$

Remark. We formulate the general hook length formula in standard terminology. Let N be a positive integer, and consider a semi-infinite checkerboard with top and left edges. A *Ferrers diagram* is a finite subset of the squares of the board which is closed under taking a unit step towards either edge. Given a Ferrers diagram with N squares, a *standard Young tableau* for this diagram is a bijection of the squares of the diagram with the integers $1,\dots,N$ such that the numbers always increase under taking a unit step away from either edge.

For each square $s = (i,j)$ in the diagram, the *hook length* h_s of s is the number of squares (i',j') in the diagram such that either $i = i', j \leq j'$ or $i \leq i', j = j'$ (including s itself). Then the number of standard Young tableaux for this diagram equals

$$\frac{N!}{\prod_s h_s}.$$

For a proof along the lines of the argument given in the previous remark, see: Kenneth Glass and Chi-Keung Ng, A simple proof of the hook length formula, *American Mathematical Monthly* **111** (2024), 700–704.

A4 The prime $p = 7$ works: choose $a = 5$ and $r = 3$, and note that $1,a,a^2$ can be rearranged to form $b_0 = 5$, $b_1 = 1$, $b_2 = 25$ satisfying the stated property.

We claim that no prime $p > 7$ works. Suppose otherwise: there exist p,a,r with $p > 7$ and $r \nmid p$ such that $1,a,\dots,a^{p-5}$ can be rearranged to form $b_0,\dots,b_{p-5}$ with $b_n \equiv b_0 + nr \pmod{p}$ for all $0 \leq n \leq p-5$. Since $r \nmid p$, $\{b_0, b_0+r, \dots, b_0+(p-5)r\}$ represents a collection of $p-4$ distinct elements of $\mathbb{Z}/p\mathbb{Z}$. It follows that all of $1,a,\dots,a^{p-5}$ are distinct mod p. In particular, $p \nmid a$; also, since $p-5 \geq \frac{p-1}{2}$, we conclude that $a^k \not\equiv 1 \pmod{p}$ for any $1 \leq k \leq \frac{p-1}{2}$. It follows that a is a primitive root mod p.

Since a is a primitive root, $a^{-3}, a^{-2}, a^{-1}, a^0, \dots, a^{p-5}$ runs through all nonzero elements of $\mathbb{Z}/p$ exactly once. On the other hand, $b_0-4r, b_0-3r, b_0-2r, b_0-r, b_0, \dots, b_0+(p-5)r$ runs through all elements of $\mathbb{Z}/p\mathbb{Z}$ exactly once. The given condition now implies that

$$\{b_0-4r, b_0-3r, b_0-2r, b_0-r\} = \{0, c, c^2, c^3\}$$

where $c = a^{-1}$; that is, $0,c,c^2,c^3$ can be rearranged to give an arithmetic sequence x_1,x_2,x_3,x_4 in $\mathbb{Z}/p\mathbb{Z}$.

If one of the two outer terms x_1 or x_4 is equal to 0, then c,c^2,c^3 can be rearranged to give a three-element arithmetic sequence. This implies that one of the following is 0: $-2c+c^2+c^3$, $c-2c^2+c^3$, $c+c^2-2c^3$. Factoring and using the fact that $c \neq 0,1$, we conclude that $c = -2$ or $c = -2^{-1}$, and the three-element arithmetic sequence is $4,-2,-8$ or $4^{-1},-2^{-1},-8^{-1}$. It is straightforward to check that in neither case can the arithmetic sequence be elongated to a four-element arithmetic sequence with 0 on one of the ends.

If one of the two inner terms x_2 or x_3 is equal to 0, then $x_3 = -x_1$ or $x_4 = -x_2$ respectively. This implies that two of c,c^2,c^3 are negatives of each other mod p. But $c \neq -c^2$ and $c^2 \neq -c^3$ since otherwise $c^3 = c$; it follows that $c^3 = -c$. But now $c^4 = 1$, whence $a^4 = 1$ and a is not a primitive root mod p. This concludes the proof.

A5 We will show that $r = 0$ (and no other value of r) minimizes the stated probability.

Note that P and Q coincide with probability 0; thus we can assume that $P \neq Q$.

First restrict P,Q to points on Ω such that the segment $\overline{PQ}$ makes an angle of θ with the y axis, where θ is a fixed number with $-\pi/2 < \theta \leq \pi/2$. By rotating the diagram by $-\theta$ around the origin, we move $\overline{PQ}$ to be a vertical line and move Δ to be centered at $(r\cos\theta, -r\sin\theta)$. In this rotated picture, P and Q are at $(9\cos\phi, \pm 9\sin\phi)$ where ϕ is chosen uniformly at random in $(0,\pi)$. Now the vertical tangent lines to the boundary of Δ, $x = r\cos\theta \pm 1$, intersect the $y > 0$ semicircle of Ω at $(9\cos\phi, 9\sin\phi)$ where $\phi = \cos^{-1}\left(\frac{r\cos\theta \pm 1}{9}\right)$. Thus the probability that $\overline{PQ}$ intersects Δ for a specific value of θ is $\frac{1}{\pi} f(r,\theta)$, where we define

$$f(r,\theta) = \cos^{-1}\left(\frac{r\cos\theta - 1}{9}\right) - \cos^{-1}\left(\frac{r\cos\theta + 1}{9}\right).$$

If we now allow θ to vary (uniformly) in $(-\pi/2, \pi/2]$, we find that the overall probability that $\overline{PQ}$ intersects Δ is

$$P(r) = \frac{1}{\pi^2}\int_{-\pi/2}^{\pi/2} f(r,\theta)\,d\theta.$$

The function $P(r)$ is differentiable with

$$P'(r) = \frac{1}{\pi^2} \int_{-\pi/2}^{\pi/2} \frac{\partial f(r,\theta)}{\partial r}\,d\theta.$$

Now

$$\frac{\partial f(r,\theta)}{\partial r} = (\cos t)\Big((80 - 2r\cos t - r^2\cos^2 t)^{-1/2} \\ -(80 + 2r\cos t - r^2\cos^2 t)^{-1/2}\Big),$$

which, for $t \in (-\pi/2, \pi/2)$, is zero for $r = 0$ and strictly positive for $r > 0$. It follows that $P'(0) = 0$ and $P'(r) < 0$ for $r \in (0, 8]$, whence $P(r)$ is minimized when $r = 0$.

A6 The determinant equals $10^{n(n-1)/2}$. To show this, we compute the corresponding determinant for the coefficients of the generic power series

$$f(x) := \sum_{n=1}^{\infty} c_n x^n, \qquad c_1 = 1,$$

with associated continued fraction

$$\frac{a_0}{x^{-1} + b_0 + \frac{a_1}{x^{-1}+b_1+\cdots}}, \qquad a_0 = 1.$$

If we truncate by replacing $a_{n+1} = 0$, we get a rational function which can be written as $\frac{A_n(x^{-1})}{B_n(x^{-1})}$ where $A_n(x), B_n(x)$ are polynomials determined by the initial conditions

$$A_{-1}(x) = 1, A_0(x) = 0, \quad B_{-1}(x) = 0, B_0(x) = 1$$

and the recurrences

$$\begin{aligned} A_{n+1}(x) &= (x + b_n)A_n(x) + a_n A_{n-1}(x) \qquad &(n > 0)\\ B_{n+1}(x) &= (x + b_n)B_n(x) + a_n B_{n-1}(x) \qquad &(n > 0). \end{aligned}$$

Since each additional truncation accounts for two more coefficients of the power series, we have

$$\frac{A_n(x^{-1})}{B_n(x^{-1})} = f(x) + O(x^{2n+1}),$$

or equivalently (since $B_n(x)$ is monic of degree n)

$$f(x)B_n(x^{-1}) - A_n(x^{-1}) = O(x^{n+1}). \tag{1}$$

We now reinterpret in the language of *orthogonal polynomials*. For a polynomial $P(x) = \sum_i P_i x^i$, define

$$\int_\mu P(x) = \sum_i P_i c_{i+1};$$

then the vanishing of the coefficient of x^{i+1} in (1) (with $n := i$) implies that

$$\int_\mu x^i B_j(x) = 0 \qquad (j < i).$$

By expanding $0 = \int_\mu x^{i-1} B_{i+1}(x)$ using the recurrence, we deduce that $\int_\mu x^i B_i(x) + a_i \int_\mu x^{i-1} B_{i-1}(x) = 0$, and so

$$\int_\mu x^i B_i(x) = (-1)^i a_1 \cdots a_i.$$

We deduce that

$$\int_\mu B_i(x)B_j(x) = \begin{cases} 0 & i \neq j \\ (-1)^i a_1 \cdots a_i & i = j. \end{cases} \tag{2}$$

In other words, for U the $n \times n$ matrix such that U_{ij} is the coefficient of x^j in $B_i(x)$, the matrix UAU^t is a diagonal matrix D with diagonal entries $D_{i,i} = (-1)^{i-1} a_1 \cdots a_{i-1}$ for $i = 1, \dots, n$. Since U is a unipotent matrix, its determinant is 1; we conclude that

$$\det(A) = \det(D) = (-1)^{n(n-1)/2} a_1^{n-1} \cdots a_{n-1}.$$

We now return to the sequence $\{c_n\}$ given in the problem statement, for which

$$f(x) = \frac{1 - 3x - \sqrt{1 - 14x + 9x^2}}{4}.$$

For

$$g(x) := \frac{1 - 7x - \sqrt{1 - 14x + 9x^2}}{2},$$

we have

$$f(x) = \frac{1}{x^{-1} - 5 - g(x)}, \quad g(x) = \frac{10}{x^{-1} - 7 - g(x)}.$$

This means that the continued fraction is periodic; in particular, $a_1 = a_2 = \cdots = -10$. Plugging into the general formula for $\det(A)$ yields the desired result. This yields the desired result.

Remark. A matrix A whose i,j-entry depends only on $i + j$ is called a *Hankel matrix*. The above computation of the determinant of a Hankel matrix in terms of continued fractions is adapted from H.S. Wall, *Analytic Theory of Continued Fractions*, Theorems 50.1 and 51.1.

The same analysis shows that if we define the sequence $\{c_n\}_{n=1}$ by $c_1 = 1$ and

$$c_n = ac_{n-1} + b\sum_{i=1}^{n-1} c_i c_{n-i} \qquad (n > 1),$$

then $a_n = -ab - b^2$, $b_n = -a - 2b$ for all $n > 0$ and so

$$\det(A) = (ab + b^2)^{n(n-1)/2};$$

the problem statement is the case $a = 3, b = 2$. The case $a = 0, b = 1$ yields the sequence of Catalan numbers; the case $a = 1, b = 1$ yields the Schröder numbers (OEIS sequence A006318).

There are a number of additional cases of Hankel determinants of interest in combinatorics. For a survey, see: A. Junod, Hankel determinants and orthogonal polynomials, *Expositiones Mathematicae* **21** (2003), 63–74.

B1 This is possible if and only if n is odd and $k = (n+1)/2$.

We first check that these conditions are necessary. If the pairs $(a_1,b_1),\dots,(a_n,b_n)$ index squares of the grid with no two in the same row or column, then each of the two sequences $a_1,\dots,a_n$ and $b_1,\dots,b_n$ is a permutation of $\{1,\dots,n\}$, and so in particular has sum $1+\cdots+n = \frac{n(n+1)}{2}$. In particular, if the selected numbers are $1,2,\dots,n$ in some order, then

$$\begin{aligned}\frac{n(n+1)}{2} &= \sum_{i=1}^n (a_i+b_i-k) \\ &= \sum_{i=1}^n a_i + \sum_{i=1}^n b_i - \sum_{i=1}^n k \\ &= \frac{n(n+1)}{2} + \frac{n(n+1)}{2} - nk\end{aligned}$$

which simplifies to $k = (n+1)/2$.

We next check that these conditions are sufficient. For this, it suffices to observe that the sequence

$$\left(1, \frac{n+1}{2}\right), \left(2, \frac{n+3}{2}\right), \dots, \left(\frac{n+1}{2}, n\right), \\ \left(\frac{n+3}{2}, 1\right), \dots, \left(n, \frac{n-1}{2}\right)$$

of grid entries equals

$$1,3,\dots,n,2,\dots,n-1.$$

We illustrate this for the case $n=5, k=3$ below; the selected entries are parenthesized.

$$\begin{pmatrix} -1 & 0 & (1) & 2 & 3 \\ 0 & 1 & 2 & (3) & 4 \\ 1 & 2 & 3 & 4 & (5) \\ (2) & 3 & 4 & 5 & 6 \\ 3 & (4) & 5 & 6 & 7 \end{pmatrix}$$

B2 No, there is no such sequence. In other words, any sequence of convex quadrilaterals with the property that any two consecutive terms are partners must be finite.

Lemma 1. *Given five positive real numbers a,b,c,d,K, there are only finitely many convex quadrilaterals with side lengths a,b,c,d in that order and area K.*

Proof. Let $PQRS$ be a convex quadrilateral with

$$\overline{PQ} = a, \overline{QR} = b, \overline{RS} = c, \overline{SP} = d.$$

Then the congruence class of $PQRS$ is uniquely determined by the length of the diagonal $f := \overline{PR}$. Moreover, as f increases, the angles $\angle RPQ$ and $\angle RPS$ are both strictly decreasing, so $\angle SPQ$ is decreasing; by the same logic, $\angle QRS$ is decreasing.

We next recall *Bretschneider's formula*: for $s = (a+b+c+d)/2$,

$$K^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\frac{\angle SPQ + \angle QRS}{2}.$$

Consequently, fixing K also fixes $\cos^2\frac{\angle SPQ+\angle QRS}{2}$, and thus limits $\angle SPQ + \angle QRS$ to one of two values. By the previous paragraph, this leaves at most two possible congruence classes for the triangle. □

Returning to our original sequence, note that any two consecutive quadrilaterals in the sequence have the same area and the same unordered list of side lengths. The latter can occur as an ordered list in at most six different ways (up to cyclic shift); for each of these, we can have only finitely many distinct congruence classes of quadrilaterals in our sequence with that area and ordered list of side lengths. We deduce that our sequence must be finite.

Remark. We give an alternate proof of the lemma using Cartesian coordinates. We first specify

$$P = (0,0), Q = (a,0).$$

For two additional points $R = (x,y), S = (z,w)$, the conditions $\overline{QR} = b$, $\overline{SP} = d$ restrict R and S to the circles

$$(x-a)^2 + y^2 = b^2, \quad z^2 + w^2 = d^2$$

respectively. Since we want a convex quadrilateral, we may assume without loss of generality that $y,w > 0$. The area of the quadrilateral is $\frac{1}{2}a(y+w)$, which we also want to fix; we may thus regard w as a function of y (possibly restricting y to a range for which $w > 0$). After splitting the semicircles on which R and S lie into two arcs each, we may also regard x and w as functions of y. It now suffices to observe that $\overline{RS}^2 = (z-x)^2 + (w-y)^2$ is a nonconstant algebraic function of y, so it takes any given value only finitely many times.

B3 Define the function

$$f(x) := \tan x - x.$$

We then have $f'(x) = \tan^2 x$. By induction on k, $f^{(k)}(x)$ is a polynomial of degree $k+1$ in $\tan x$ with leading coefficient $k!$ and all coefficients nonnegative. In particular, on each of the intervals

$$I_n := \left(n\pi, n\pi + \frac{\pi}{2}\right) \qquad (n = 0,1,\dots),$$

$\tan x$ is positive and so $f^{(k)}(x)$ is positive for each $k \geq 1$; replacing k with $k+1$, we deduce that each $f^{(k)}(x)$ is strictly increasing on I_n for $k \geq 0$.

We now analyze f more closely on I_n. As $x \to n\pi^+$ for $n > 0$, $f(x)$ tends to $f(n\pi) = -n\pi < 0$; by contrast, as $x \to 0^+$, $f(x)$ tends to 0 via positive values. In either case, as $x \to (n\pi + \frac{\pi}{2})^-$, $f(x) \to \infty$. Since $f(x)$ is strictly increasing on I_n, we deduce using the intermediate value theorem that:

- $f(x)$ has no zero in I_0;
- for $n > 0$, $f(x)$ has a unique zero in I_n.

Since $f(x)$ also has no zero between I_n and I_{n+1} (as it takes exclusively negative values there), we deduce that

$$n\pi < r_n < n\pi + \frac{\pi}{2}.$$

This already suffices to prove the claimed lower bound: since $f(r_n+\pi) = -\pi < 0$ and f is strictly increasing on I_{n+1}, the quantity $\delta := r_{n+1} - (r_n+\pi)$ is positive.

To prove the upper bound, note that for $k \geq 1$, for $0 < x < n\pi + \frac{\pi}{2} - r_n$, we have

$$\begin{aligned} f^{(k)}(x) &\geq f^{(k)}(r_n+\pi) \\ &= f^{(k)}(r_n) \\ &\geq k! r_n^{k+1} \\ &> k! n^{k+1}\pi^{k+1}. \end{aligned}$$

For each $k \geq 2$, we may apply the mean value theorem with remainder to deduce that for x in the same range,

$$f(r_n+\pi+x) \geq f(r_n+\pi) + \sum_{i=1}^k f^{(i)}(r_n+\pi)\frac{x^i}{i!}.$$

Taking the limit as $k \to \infty$ yields

$$\begin{aligned} f(r_n+\pi+x) &\geq f(r_n+\pi) + \sum_{i=1}^\infty f^{(i)}(r_n+\pi)\frac{x^i}{i!} \\ &> -\pi + \sum_{i=1}^k n^{i+1}\pi^{i+1}x^i \\ &> -\pi + \frac{n^2\pi^2 x}{1-n\pi x}; \end{aligned}$$

taking $x = \delta$ yields

$$0 > -\pi + n\pi\left(\frac{1}{1-n\pi\delta} - 1\right)$$

and so $\delta < \frac{1}{n(n+1)\pi}$ as desired.

Remark. There is a mild subtlety hidden in the proof: if one first bounds the finite sum as

$$f(r_n+\pi+x) > -\pi + \sum_{i=1}^k n^{i+1}\pi^{i+1}x^i$$

and then takes the limit as $k \to \infty$, the strict inequality is not preserved. One way around this is to write $f''(r_n) = 2r_n + 2r_n^3$, retain the extra term $r_n x^2$ in the lower bound, take the limit as $k \to \infty$, and then discard the extra term to get back to a strict inequality.

Remark. The slightly weaker inequality $\delta < \frac{1}{n^2\pi}$ follows at once from the inequality

$$f'(r_n+\pi) = f'(r_n) = \tan^2 r_n = r_n^2 > n^2\pi^2$$

plus the mean value theorem.

B4 The limit equals $\frac{1-e^{-2}}{2}$.

We first reformulate the problem as a Markov chain. Let v_k be the column vector of length n whose i-th entry is the probability that $a_{n,k} = i$, so that v_0 is the vector $(1,0,\dots,0)$. Then for all $k \geq 0$, $v_{k+1} = Av_k$ where A is the $n \times n$ matrix defined by

$$A_{ij} = \begin{cases} \frac{1}{n} & \text{if } i = j \\ \frac{j-1}{n} & \text{if } i = j-1 \\ \frac{n-j}{n} & \text{if } i = j+1 \\ 0 & \text{otherwise.} \end{cases}$$

Let w be the row vector $(1,\dots,n)$; then the expected value of $a_{n,k}$ is the sole entry of the 1×1 matrix $wv_k = wA^k v_0$. In particular, $E(n) = wA^n v_0$.

We compute some left eigenvectors of A. First,

$$w_0 := (1,\dots,1)$$

satisfies $Aw_0 = w_0$. Second,

$$\begin{aligned} w_1 &:= (n-1, n-3, \dots, 3-n, 1-n) \\ &= (n-2j+1\colon j = 1,\dots,n) \end{aligned}$$

satisfies $Aw_1 = \frac{n-2}{n}w_1$: the j-th entry of Aw_i equals

$$\begin{aligned} &\frac{j-1}{n}(n+3-2j) + \frac{1}{n}(n+1-2j) + \frac{n-j}{n}(n-1-2j) \\ &= \frac{n-2}{n}(n-2j+1). \end{aligned}$$

By the same token, we obtain

$$w = \frac{n+1}{2}w_0 - \frac{1}{2}w_1;$$

we then have

$$\begin{aligned} \frac{E(n)}{n} &= \frac{n+1}{2n}w_0A^nv_0 - \frac{1}{2n}w_1A^nv_0 \\ &= \frac{n+1}{2n}w_0v_0 - \frac{1}{2n}\left(1-\frac{2}{n}\right)^n w_1v_0 \\ &= \frac{n+1}{2n} - \frac{n-1}{2n}\left(1-\frac{2}{n}\right)^n. \end{aligned}$$

In the limit, we obtain

$$\begin{aligned} \lim_{n\to\infty}\frac{E(n)}{n} &= \frac{1}{2} - \frac{1}{2}\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n \\ &= \frac{1}{2} - \frac{1}{2}e^{-2}. \end{aligned}$$

Remark. With a bit more work, one can show that A has eigenvalues $\frac{n-2j}{n}$ for $j = 0,\dots,n-1$, and find the corresponding left and right eigenvectors.

B5 For convenience, we extend the problem to allow nonnegative values for k and m.

Let $R(n,k)$ denote the number of subsets of $\{1,...,n\}$ of size k where repetitions are allowed. The "sticks and stones" argument shows that

$$R(n,k) = \binom{n+k-1}{k}:$$

there is a bijection of these subsets with linear arrangements of k (unlabeled) sticks and $z-1$ (unlabeled) stones, where we recover the subset by counting the number of stones to the left of each stick.

Let $f_{k,m}(n) := \sum_{z=1}^{n} R(z,k)R(z,m)$. It is known that for any positive integer k, the sum of the k-th powers of all positive integers less than or equal to n is a polynomial in n (given explicitly in terms of Bernoulli numbers via Faulhaber's formula); hence $f_{k,m}(n)$ is a polynomial in n. We wish to show that this polynomial has nonnegative coefficients.

Using the recursion for binomial coefficients, we obtain

$$\begin{aligned}
R(n,k)R(n,m) &= f_{k,m}(n) - f_{k,m}(n-1) \\
&= \sum_{z=1}^{n} R(z,k)R(z,m) - R(z-1,k)R(z-1,m) \\
&= \sum_{z=1}^{n} R(z,k)R(z,m) - R(z-1,k)R(z,m) \\
&\quad + R(z-1,k)R(z,m) - R(z-1,k)R(z-1,m) \\
&= \sum_{z=1}^{n} R(z,k-1)R(z,m) + R(z-1,k)R(z,m-1) \\
&= \sum_{z=1}^{n} R(z,k-1)R(z,m) \\
&\quad + (R(z,k) - R(z,k-1))R(z,m-1) \\
&= f_{k-1,m} + f_{k,m-1} - f_{k-1,m-1}.
\end{aligned}$$

It follows from the latter equation (replacing the index m by $m+1$) that

$$f_{k,m}(n) = R(n,k)R(n,m+1) + f_{k-1,m}(n) - f_{k-1,m+1}(n); \quad (3)$$

this can also be recovered by applying Abel summation (summation by parts) to $\sum_{z=1}^{n} R(z,k)R(z,m)$.

Using (3), we can evaluate $f_{k,m}$ by induction on k: for the first few values we obtain

$$\begin{aligned}
f_{0,m}(n) &= R(n,m+1) \\
f_{1,m}(n) &= R(n,1)R(n,m+1) + R(n,m+1) - R(n,m+2) \\
&= R(n,m+1)((m+1)n+1)/(m+2) \\
&= R(n,m+1)\frac{R(m+1,1)R(n,1)+1}{m+2}
\end{aligned}$$

and similarly

$$\begin{aligned}
f_{2,m}(n) &= R(n,m+1)(R(m+1,2)R(n,2) + R(m+1,1)R(n,1) \\
&\quad + R(m+1,0)R(n,0))/R(m+2,2).
\end{aligned}$$

This leads us to conjecture that

$$f_{k,m}(n) = \frac{R(n,m+1)}{R(m+2,k)} \sum_{i=0}^{k} R(m+1,i)R(n,i), \qquad (4)$$

which we prove by induction on k; the cases $k \leq 2$ are covered by our prior work (in fact only $k=0$ is needed). Given (4) with k replaced by $k-1$, we apply (3) to obtain

$$\begin{aligned}
&f_{k,m}(n) \\
&= R(n,k)R(n,m+1) + \frac{R(n,m+1)}{R(m+2,k-1)} \sum_{i=0}^{k-1} R(m+1,i)R(n,i) \\
&\quad - \frac{R(n,m+2)}{R(m+3,k-1)} \sum_{i=0}^{k-1} R(m+2,i)R(n,i) \\
&= \frac{R(n,m+1)}{R(m+2,k)} \sum_{i=0}^{k} R(m+1,i)R(n,i)
\end{aligned}$$

yielding (4) as written.

Since $R(n,i) = n(n+1)(n+2)\cdots(n+i-1)/i!$ clearly has positive coefficients for all i, the explicit formula (4) implies that $f_{k,m}(n)$ also has positive coefficients for all k and m.

B6 The claim holds with $c = -\frac{1}{2}$. Set $t := 1/(1-x)$, so that $x = 1 - 1/t$ and

$$-\frac{1}{t} - \frac{1}{t^2} \leq \log x \leq -\frac{1}{t}.$$

Set also $m := \lfloor t \rfloor$.

Suppose first that $a > -\frac{1}{2}$. Then

$$\begin{aligned}
F_a(x)e^{-t} &= \sum_{n=1}^{\infty} n^a e^{2n-t} x^{n^2} \\
&\geq \sum_{n=1}^{\infty} n^a e^{2n-t-n^2/t-n^2/t^2} \\
&= \sum_{n=1}^{\infty} n^a e^{-n^2/t^2} e^{-t(1-n/t)^2}.
\end{aligned}$$

If we restrict the sum to the range $t < n < t + \sqrt{t}$, we may bound the summand from below by ct^a for some $c > 0$ independent of t; we then have $F_a(x)e^{-t} > ct^{a+1/2}$ and this tends to ∞ as $t \to \infty$.

Suppose next that $a < -\frac{1}{2}$. Then

$$\begin{aligned}
F_a(x)e^{-t} &= \sum_{n=1}^{\infty} n^a e^{2n-t} x^{n^2} \\
&\leq \sum_{n=1}^{\infty} n^a e^{-t(1-n/t)^2}.
\end{aligned}$$

Fix ε such that $a + \varepsilon < -\frac{1}{2}$. For the summands with $t - t^{1/2+\varepsilon} < n < t + t^{1/2+\varepsilon}$, we may similarly bound the summand from above by ct^a for some $c > 0$; this range of the sum is then dominated by $ct^{a+1/2+\varepsilon}$ for some $c > 0$. For the summands with $n < t - 2\sqrt{t}$, we may bound the summand from above by $n^a e^{-t^{2\varepsilon}}$; this range of the sum is then dominated by $te^{-t^{2\varepsilon}}$. For the summands with $n > t - 2\sqrt{t}$, we may again bound the summand from above by $n^a e^{-t^{2\varepsilon}}$; this range of the sum is then dominated by $t^{a+1} e^{-t^{2\varepsilon}}$. Since all three bounds tends to 0 as $t \to \infty$, so then does $F_a(x)e^{-t}$.