Review Exercises

The following exercises are meant to review various areas we have mentioned before. In fact these are all from the IMC, with problem numbers taken from Mike Daas's IMC.pdf.

1. Problem 12.65 (IMC 2002). A total of 200 students participated in a math Olympiad that featured 6 problems. Every problem was solved by at least 120 students. Prove that there are two students who together solved all the problems.

2. Problem 6.35 (IMC 1999). We throw a fair six-sided dice n times. What is the probability that the sum of the values is divisible by 5?

3. Problem 11.23 (IMC 2007). Let $p(x) \in \mathbb{Z}[x]$ be a polynomial of degree 2. Suppose $p(n)$ is divisible by 5 for every integer n. Prove that all coefficients of p are divisible by 5.

4. Problem 11.21 (IMC 2005). Find all polynomials of degree n whose coefficients are a permutation of the numbers $\{0, 1, \ldots, n\}$ and all of whose roots are rational numbers.

5. Problem 3.24 (IMC 1998). Let $V = \mathbb{R}^{10}$ and let $U_1 \subseteq U_2 \subseteq V$ be subspaces with $\dim(U_1) = 3$ and $\dim(U_2) = 6$. Let E be the space of linear maps $T : V \to V$ such that $T(U_1) \subseteq U_1$ and $T(U_2) \subseteq U_2$. Determine $\dim(E)$.

6. Problem 5.70 (IMC 2003). Determine the set of all pairs (a, b) of positive integers for which the set $\mathbb{Z}^+$ of positive integers can be decomposed into two sets A and B such that $a \cdot A = b \cdot B$.