Universiteit Leiden

MATHEMATICS

NOTES

A Brief IMC Training

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0 Introduction

This document constitutes a very brief introduction into the world of university level mathematics olympiads. In terms of prestige they may not quite rival those aimed at high school students, but this does not take away from the enjoyment of toying with some slightly more advanced mathematical notions in a fresh and creative way. This introduction is irresponsibly brief and as such, we have had to focus on only those ideas that we deemed most prevalent at international mathematics olympiads, particularly at the IMC. It is the author's hope that the reader will find enjoyment in solving these problems, and that at least once in their mathematical career, one of these tricks may prove useful beyond the scope of mere olympiads.

After a brief section outlining some theory and highlighting a few selected tips and tricks, the reader should not feel overwhelmed by the sheer amount of problems that await them in each chapter. They have been compiled from various different sources, but most prominently they are taken directly from the IMC, the competition that this note primarily aims to prepare its readers for. Credit for other problems goes to Iris Smit and Josha Box and their own Dutch IMC training syllabus that is often used in Amsterdam, but also to Julian Lyczak and his "Wiskundewedstrijdtraining" for students in Utrecht. With this document, it was my intention to bring these Dutch materials to a more international audience.

The IMC problems are included in chronological order and may therefore vary in difficulty substantially from one to the next. Each chapter is also accompanied by a smaller set of problems that illustrate various concepts discussed in the theory preceding them, which we would recommend those new to the sport to attempt first before diving head first in the deep end. Namely, IMC problems can be quite challenging, and even though some of the other problems in this set are equally difficult, at least the first few of each chapter should be considerably more approachable than most of what one would find at the IMC. The most challenging problems, according to the author's judgment at a glance, have been marked by an asterisk $(*)$, but please keep in mind that some of these may have been placed inappropriately. Some IMC problems, albeit relevant to the topics, have been omitted from this set if the author deemed it in the reader's best interest to shield them from the technicalities that would be difficult to avoid when seriously giving these problems a proper attempt.

Should the reader find any typos, errors or other misprints, they are most welcome to contact the author at dutchmikedaas@gmail.com. Any feedback would be greatly appreciated. It is the author's hope that these notes will grow out to become a useful tool for those new to the world of university level olympiads, and that these competitions will bring them as much joy as it has brought yours truly.

Contents

1 Analysis I

Real analysis problems are very common at the IMC. The fundamentals of this interesting field of study contain many cute yet powerful lemmas that can be exploited in non-trivial ways to solve some pleasantly tricky problems, often without the need for tedious ϵ, δ-proofs. Due to the scope of this topic, we split its treatment into many separate sections; the first two will cover the theory of sequences and series, whereas the latter two will focus on functions, continuity and integration. Let us start with sequences first.

1.1 Real sequences

Let us start by recalling a handful of topological notions about subsets of the real numbers. We assume the reader to be familiar with the standard topology on $\mathbb{R}$.

Definition 1.1. Let $S \subset \mathbb{R}$. We say some $x \in S$ is an *interior point* if there exists some open neighbourhood $x \in U \subset S$. We say some $x \in S$ is *isolated* if for some open $U \ni x$ we have $U \cap S = \{x\}$.

We say that some $x \in \mathbb{R}$ is a *limit point* of S if for every open $U \ni x$, the set $U \cap S$ is infinite. We say some $x \in \mathbb{R}$ is an *edge point* of S if for every open $U \ni x$, the sets $U \cap S$ and $U \cap (\mathbb{R} \setminus S)$ are both non-empty.

Definition 1.2. We say some $x \in \mathbb{R}$ is a limit point of a sequence $(a_n)_{n \in \mathbb{N}}$ if some subsequence of (a_n) converges to x. The infimum of this set is denoted by $\liminf a_n$, and the supremum of this set is denoted $\limsup a_n$. Recall that a sequence converges precisely when these two notions agree.

The following elementary results are sometimes very useful when proving statements about sequences.

Lemma 1.3. *Every sequence of real numbers contains a monotone subsequence.*

Lemma 1.4. *Every bounded monotonous sequence of real numbers is convergent.*

Proposition 1.5. (Bolzano-Weierstraß) *Every bounded sequence in* $\mathbb{R}$, *or even* $\mathbb{R}^n$, *contains a convergent subsequence.*

1.2 Sequences and functions

Even though a thorough treatment of functions is left to the next chapter, it is impossible to omit them completely from any comprehensive treatment of sequences and series. Often sequences are recursively defined, which one can generally write as $a_n = f(a_{n-1}, \dots, a_{n-k})$. We recall the following definition. which is typically more useful than the standard ϵ, δ-definition.

Definition 1.6. A function $f : U \to \mathbb{R}$ for some subset $U \subset \mathbb{R}$ is called *continuous* at some $x \in U$ if for every sequence (a_n) of elements in U that converges to x, the sequence $f(a_n)$ converges to $f(x)$. We say that f is continuous if it is so at all $x \in U$.

This definition makes it very easy to determine the possible limit values of a recursively defined function.

Proposition 1.7. *Let (a_n) be a recursively defined sequence by $a_n = f(a_{n-1}, \dots, a_{n-k})$ for some function f. Suppose that (a_n) converges to some limit L. Then $L = f(L, \dots, L)$.*

However, it does not show that such a sequence should even converge in the first place. The following theorem sometimes helps with this.

Proposition 1.8. (Banach's Fixed Point Theorem) *Let X be a closed connected subset of $\mathbb{R}$. Suppose that f is a contraction; in other words, there exists some $0 < c < 1$ such that*

$$|f(x) - f(y)| \leqslant c|x - y| \quad \textit{for all } x, y \in X.$$

Then there exists a unique solution x^* to the equation $f(x) = x$. In addition, for any $x_0 \in X$, the sequence defined by $x_n = f(x_{n-1})$ for $n \geqslant 1$ converges to x^*.

By definition of the Riemann integral, we know that for an integrable function f, a sum of the form

$$\sum_{i=1}^{n} f(x_i)(a_{i+1} - a_i) \quad \text{converges to} \quad \int_a^b f(x)dx$$

if $x_i \in (a_i, a_{i+1})$ for some elements $a = a_0 < \dots < a_{n+1} = b$ and the partition length goes to zero as $n \to \infty$. The definition of Riemann sums does not make it easy to actually compute the integral, so sometimes a sum is cleverly disguised as one. This way, its value can only be determined by recognising the integral that it secretly defines. However, these can be very difficult to spot when hidden well.

1.3 Inequalities

The real numbers enjoy the property of being totally ordered. This allows them to satisfy many interesting and useful inequalities, the most fundamental of which is the following:

$$x^2 \geqslant 0 \quad \text{for all } x \in \mathbb{R}\text{, and} \quad x^2 = 0 \iff x = 0.$$

Another very famous and often useful inequality is the following.

Theorem 1.9. *Let $a_1, \ldots, a_n > 0$ be positive real numbers and define for $p \neq 0$,*

$$M_p = \left(\frac{x_1^p + \ldots + x_n^p}{n}\right)^{1/p}, \quad \textit{and} \quad M_0 = \sqrt[n]{a_1 \cdots a_n}.$$

Then if $p < q$, we have $M_p \leqslant M_q$. Equality holds if and only if all a_i are equal. In particular, we have

$$\frac{a_1 + \ldots + a_n}{n} \geqslant \sqrt[n]{a_1 \cdots a_n} \geqslant \frac{n}{\frac{1}{a_1} + \ldots + \frac{1}{a_n}}.$$

An even more general version using weights can be deduced from *Jensen's Inequality*, which is discussed in the next chapter. The following is also very famous.

Theorem 1.10. (Cauchy-Schwarz) *Let X be an inner product space and let $u, v \in X$. Then $|u \cdot v| \leqslant \|u\| \cdot \|v\|$. Equality only holds when u and v are linearly dependent. In particular, for any $x, y \in \mathbb{R}^n$, it holds that*

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leqslant \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right).$$

The following inequality may seem obvious after digesting its content, but it can be used in non-trivial ways.

Proposition 1.11. (Rearrangement Inequality) *Let $x_1 \geqslant \ldots \geqslant x_n$ and $y_1 \geqslant \ldots \geqslant y_n$ be real numbers. Then for any $\sigma \in S_n$, we have*

$$x_1y_1 + \ldots + x_ny_n \geqslant x_1y_{\sigma(1)} + \ldots x_ny_{\sigma(n)} \geqslant x_1y_n + \ldots x_ny_1.$$

This can be used to deduce the following.

Theorem 1.12. (Tsjebychev's Inequality) *Let $x_1 \geqslant \ldots \geqslant x_n$ and $y_1 \geqslant \ldots \geqslant y_n$ be real numbers. Then*

$$\frac{x_1y_1 + \ldots + x_ny_n}{n} \geqslant \frac{x_1 + \ldots + x_n}{n} \cdot \frac{y_1 + \ldots + y_n}{n} \geqslant \frac{x_1y_n + \ldots + x_ny_1}{n}.$$

The following inequalities help deal with arbitrary exponents.

Proposition 1.13. (Bernoulli's Inequality) *Let $r \geqslant 1$ and let $x > -1$. Then*

$$(1 + x)^r \geqslant 1 + rx.$$

Proposition 1.14. (Schur's Inequality) *Let $x, y, z \geqslant 0$ and $t > 0$. Then*

$$x^t(x - y)(x - z) + y^t(y - z)(y - x) + z^t(z - x)(z - y) \geqslant 0.$$

1.4 Examples

Example 1.15. Prove that

$$\sqrt{6+\sqrt{6+\sqrt{6+\ldots}}}$$

converges and determine its limit.

Solution: We formalise the problem first: let $a_1, a_2, \ldots$ be the sequence defined by

$$a_1 = \sqrt{6} \quad \text{and} \quad a_{n+1} = \sqrt{6+a_n}$$

for all $n \geqslant 1$. We are then tasked to show that $\lim_{n\to\infty} a_n$ converges and to find its value. To show that it converges, we will show that it is both increasing and bounded above. Indeed, it quickly follows that

$$a_n \leqslant 3 \implies a_{n+1} = \sqrt{6+a_n} \leqslant \sqrt{9} = 3,$$

so the sequence is bounded by induction. To show it is increasing, we note that

$$a_{n+1} \geqslant a_n \iff 6 + a_n \geqslant a_n^2 \iff a_n \in [-2, 3].$$

Convergens follows. Since the function $f(x) = \sqrt{6+x}$ is continuous for $x > -6$, it follows that the limit L of the sequence must satisfy $f(L) = L$; i.e. it must hold that $L = \sqrt{6+L}$. It readily follows that $L = 3$. △

Example 1.16. Determine

$$\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n-1} \right).$$

Solution: Define a function f by setting $f(x) = 1/n$ if $x \in [n, n+1]$ as n ranges over the positive integers. Then note that on the interval $[n, n+1]$, we have that

$$\left| f(x) - \frac{1}{x} \right| \leqslant \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.$$

One may then compute that

$$\frac{1}{n+1} \geqslant \int_n^{2n} \left| f(x) - \frac{1}{x} \right| dx = \sum_{k=n}^{2n-1} \frac{1}{k} - \int_n^{2n} \frac{1}{x} dx = \sum_{k=n}^{2n-1} \frac{1}{k} - \log(2).$$

If we now let $n \to \infty$, the error tends to zero and we find that the limit equals $\log(2)$, completing the proof. △

1.5 Exercises

Problem 1.1. *Prove that for all* $x, y, z \in \mathbb{R}$*,*

$$x^2 + y^2 + z^2 \geqslant xy + yz + zx.$$

Problem 1.2. *Let* (x_n) *be the sequence defined by* $x_0 = 1$ *and* $x_{n+1} = \sin(x_n)$*. Similarly, let* (y_n) *be defined by* $y_0 = 1$ *and* $y_{n+1} = \cos(y_n)$*. Show that both sequences converge.*

Problem 1.3. *Consider the sequences*

$$a_n = \frac{\operatorname{lcm}(1, 2, \ldots, n)}{n!} \quad \textit{and} \quad b_n = \frac{\operatorname{lcm}(1, 2, \ldots, n-1)}{\operatorname{lcm}(1, 2, \ldots, n)}.$$

Do any of these sequences converge? If so, what is their limit?

Problem 1.4. *Let* $a, b > 0$ *and let* $n \in \mathbb{N}$*. Show that*

$$\frac{a^n + b^n}{2} \geqslant \left(\frac{a + b}{2}\right)^n.$$

Problem 1.5. *Let* $a, b > 0$*. Now define two sequences* (a_n) *and* (b_n) *by setting* $a_0 = a$ *and* $b_0 = b$*, and further*

$$a_n = \frac{a_{n-1} + b_{n-1}}{2} \quad \textit{and} \quad b_n = \sqrt{a_{n-1} b_{n-1}} \quad \textit{for all } n \geqslant 1.$$

Show that a_n *and* b_n *converge to the same limit.*

Problem 1.6. *Let* (a_n) *be the sequence given by* $a_1 = \sqrt{2}$ *and* $a_{n+1} = \sqrt{2}^{a_n}$ *for all* $n \geqslant 1$*. Prove that* (a_n) *converges and determine its limit.*

Problem 1.7. *Prove that*

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \ldots}}}} = 3.$$

Problem 1.8. *Let* (a_n) *be a sequence of real numbers satisfying* $(2 - a_n)a_{n+1} = 1$ *for all* $n \geqslant 1$*. Show* (a_n) *converges and determine its limit.*

Problem 1.9. *Prove that*

$$\sqrt{7 - \sqrt{7 + \sqrt{7 - \sqrt{7 + \ldots}}}}$$

converges and determine its limit.

Problem 1.10. *Show that the sequence given by*

$$a_n = \sqrt{1 + \sqrt{2 + \sqrt{3 + \ldots + \sqrt{n}}}}$$

is convergent.

Problem 1.11. *For any* $n \geqslant 1$, *define*

$$a_n = \frac{1 \cdot 4 \cdot 7 \cdots (3n-2)}{2 \cdot 5 \cdot 8 \cdots (3n-1)}.$$

Prove that

$$\frac{1}{\sqrt{3n+1}} \leqslant a_n \leqslant \frac{1}{\sqrt[3]{3n+1}}.$$

Problem 1.12. *Let* $d \in \mathbb{R}$. *For each positive integer* m, *define the sequence* $a_m(n)$ *by setting* $a_m(0) = d \cdot 2^{-m}$ *and* $a_m(n+1) = a_m(n)^2 + 2a_m(n)$ *for all* $n \geqslant 0$. *Determine the limit given by* $\lim_{n\to\infty} a_n(n)$.

Problem 1.13. *Let* $\alpha \in \mathbb{R}$ *be given and define the sequence* (a_n) *by* $a_0 = \alpha$, $a_1 = 1$ *and further* $a_{n+2} = |a_{n+1} - a_n|$ *for all* $n \geqslant 0$. *For which choices of* α *does this sequence contain repeats? Alternatively, for which choices of* α *does the sequence eventually become periodic?*

Problem 1.14. *Let* $\alpha > 0$. *Prove that*

$$\lim_{n\to\infty} \left(1 + \frac{\alpha}{n}\right)^n = e^{\alpha}.$$

Problem 1.15. *Does the sequence* $a_n = \tan(\pi\sqrt{n^2 - n})$ *converge?*

Problem 1.16. *Let* $(a_n)_{n\geqslant 0}$ *be a sequence satisfying* $a_{n+1} = 4a_n(1-a_n)$ *for all* $n \geqslant 0$. *How many such sequences satisfy* $a_{2023} = a_0$?

Problem 1.17. *Let* $m, n \in \mathbb{N}$. *Determine*

$$\lim_{x\to 0} \frac{\cos(x)^{1/m} - \cos(x)^{1/n}}{x^2}.$$

Problem 1.18. *Let* $x_1, \ldots, x_n \in \mathbb{R}$ *and let* $y_1, \ldots, y_n > 0$. *Prove that*

$$\frac{x_1^2}{y_1} + \ldots \frac{x_n^2}{y_n} \geqslant \frac{(x_1 + \ldots x_n)^2}{y_1 + \ldots y_n}.$$

Problem 1.19. *Prove that for any integer* $n > 1$, *it holds that*

$$1 \cdot 3 \cdot 5 \cdots (2n-1) < n^n.$$

Problem 1.20. *Prove that for all* $a, b, c > 0$, *it holds that*

$$\frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} \leqslant \frac{a+b+c}{2}.$$

Problem 1.21. *Prove that for all* $x, y, z \in \mathbb{R}$, *it holds that*

$$x^4 + y^4 + z^2 \geqslant \sqrt{8}xyz.$$

Problem 1.22. *Prove that for all* $x, y, z \in \mathbb{R}$*, it holds that*

$$x^2 + y^2 + z^2 \geqslant x\sqrt{y^2 + z^2} + y\sqrt{x^2 + z^2}.$$

When does equality hold?

Problem 1.23. *Let* $x, y, z \geqslant 0$ *satisfy* $(1 + x)(1 + y)(1 + z) = 8$*. Prove that* $xyz \leqslant 1$*.*

Problem 1.24. *Prove that for all* $x, y, z, w > 0$*, it holds that*

$$\frac{1}{x} + \frac{1}{y} + \frac{4}{z} + \frac{16}{w} \geqslant \frac{64}{x + y + z + w}.$$

Problem 1.25. *Prove that for any* $x, y \geqslant 0$*, it holds that*

$$x^4 + y^4 + 8 \geqslant 8xy.$$

Problem 1.26. *Let* $m \in \mathbb{N}$ *and let* $a, b > 0$*. Prove that*

$$\left(1 + \frac{a}{b}\right)^m + \left(1 + \frac{b}{a}\right)^m \geqslant 2^{m+1}.$$

Problem 1.27. *Let* $m, n \in \mathbb{N}$ *and* $x, y \geqslant 0$*. Prove that*

$$x^m y^n \leqslant \left(\frac{mx + ny}{m + n}\right)^{m+n}.$$

Problem 1.28. *Let* $n \geqslant 2$ *be an integer and* $a > 1$*. Prove that*

$$a^n - 1 > n\left(a^{\frac{n+1}{2}} - a^{\frac{n-1}{2}}\right).$$

Problem 1.29. *Let* A*,* B *and* C *be the angles of some triangle. Prove that*

$$\sin(A) + \sin(B) + \sin(C) \geqslant \frac{3\sqrt{3}}{2}.$$

Problem 1.30. *Let* $a_1, a_2, a_3 > 0$*. Prove that*

$$\frac{a_1^2 + a_2^2 + a_3^2}{a_1^3 + a_2^3 + a_3^3} \geqslant \frac{a_1^3 + a_2^3 + a_3^3}{a_1^4 + a_2^4 + a_3^4}.$$

Problem 1.31. *Let* $a, b, p \in \mathbb{R}$ *with* $0 \leqslant p \leqslant 1$*. Prove that*

$$|a + b|^p \leqslant |a|^p + |b|^p.$$

Problem 1.32. *Let* $n > 8$*. Which of* $\sqrt{n}^{\sqrt{n+1}}$ *and* $\sqrt{n+1}^{\sqrt{n}}$ *is bigger?*

Problem 1.33. *Let* $0 \leqslant p, q, r \leqslant 1$*. Prove that*

$$(p - q)(q - r)(r - p) < \frac{8}{27}.$$

Problem 1.34. *Let* $a, b, c, A, B, C \in \mathbb{R}$ *with* $a, A \neq 0$. *Suppose that*

$$|ax^2 + bx + c| \leqslant |Ax^2 + Bx + C|$$

for all $x \in \mathbb{R}$. *Prove that*

$$|b^2 - 4ac| \leqslant |B^2 - 4AC|.$$

Problem 1.35. *Let* $x_1, \ldots x_n$ *be a permutation of the set* $\{1, 2, \ldots, n\}$. *Determine the maximal value of*

$$x_1x_2 + x_2x_3 + \ldots x_{n-1}x_n + x_nx_1.$$

Problem 1.36. *Prove that for any* $n \in \mathbb{N}$ *it holds that*

$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.$$

Problem 1.37. $(*)$ *Let* $a \neq 1$ *be a positive real number. Determine*

$$\lim_{x\to\infty} \left(\frac{a^x - 1}{x(a-1)}\right)^{1/x}.$$

Problem 1.38. $(*)$ *Determine*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}}\right).$$

Problem 1.39. $(*)$ *Determine*

$$\lim_{k\to\infty} \frac{1}{k} \int_0^k (1 + \sin(2x))^{1/x} dx.$$

Problem 1.40. $(*)$ *Determine the limit*

$$\lim_{n\to\infty} \left(\left(\frac{1}{n}\right)^1 \left(\frac{2}{n}\right)^2 \cdots \left(\frac{n}{n}\right)^n\right)^{1/n^2}.$$

Problem 1.41. $(*)$ *Given* $x \in \mathbb{R}$, *determine the value of the limit*

$$\lim_{n\to\infty} \cos\frac{x}{2} \cdot \cos\frac{x}{4} \cdots \cos\frac{x}{2^n}.$$

Problem 1.42. $(*)$ *Show that*

$$\lim_{n\to\infty} \frac{1}{n} \left(\frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdots (2n-1)}\right)^2 = \pi.$$

Problem 1.43. $(*)$ *Let* $a_1, \ldots, a_n \geqslant 0$. *Prove that*

$$(1 + a_1) \cdots (1 + a_n) \geqslant (1 + \sqrt[n]{a_1 \cdots a_n})^n.$$

Problem 1.44. $(*)$ *Consider* $a_1, \ldots, a_n \geqslant 0$ *such that* $a_1 + \ldots + a_n = 1$. *Prove that*

$$(a_1 + a_2)(a_1 + a_2 + a_3) \cdots (a_1 + a_2 + \ldots + a_{n-1}) \geqslant 4^{n-1} a_1 \cdots a_n.$$

Problem 1.45. $(*)$ *Let* $a_1 < a_2 < \ldots < a_{2n+1}$ *be real numbers. Prove that*

$$\sqrt[n]{a_1 - a_2 + a_3 - \ldots - a_{2n} + a_{2n+1}} \geqslant \sqrt[n]{a_1} - \sqrt[n]{a_2} + \sqrt[n]{a_3} - \ldots - \sqrt[n]{a_{2n}} + \sqrt[n]{a_{2n+1}}.$$

Problem 1.46. $(*)$ *Let* $n > 1$ *and* $x_1, \ldots x_n > 0$ *such that* $x_1 + \ldots x_n = 1$. *Prove that*

$$\frac{x_1}{\sqrt{1 - x_1}} + \ldots \frac{x_n}{\sqrt{1 - x_n}} \geqslant \frac{\sqrt{x_1} + \ldots + \sqrt{x_n}}{\sqrt{n - 1}}.$$

Problem 1.47. $(*)$ *Let* a, b, c *be the sides of a triangle. Prove that*

$$\frac{3}{2} \leqslant \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \leqslant 2.$$

Problem 1.48. $(*)$ *Let* $a_1, \ldots a_n \geqslant 0$ *and* $b_1, \ldots b_n \geqslant 0$. *Prove that*

$$(a_1 \cdots a_n)^{1/n} + (b_1 \cdots b_n)^{1/n} \leqslant \big((a_1 + b_1) \cdots (a_n + b_n)\big)^{1/n}.$$

1.6 IMC Problems

Problem 1.49. (1996)$(*)$ *Let* (a_n) *be the sequence defined by* $a_1 = 1$ *and*

$$a_n = \frac{1}{n} \sum_{k=1}^{n-1} a_k a_{n-k}$$

for all $n \geqslant 2$. *Show that* $2/3 \leqslant \limsup |a_n|^{1/n} < 1/\sqrt{2}$.

Problem 1.50. (1996)$(*)$ *Prove that for every sequence* (a_n) *of positive real numbers with* $\sum_{n=1}^{\infty} a_n < \infty$, *we have*

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n.$$

Problem 1.51. (1999) *Let* $x_1, \ldots, x_n \geqslant -1$ *satisfy* $\sum_{i=1}^{n} x_i^3 = 0$. *Prove that* $\sum_{i=1}^{n} x_i \leqslant n/3$.

Problem 1.52. (2000) *Let* (x_i) *be a decreasing sequence of positive numbers. Prove that*

$$\left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \leqslant \sum_{i=1}^{n} \frac{x_i}{\sqrt{i}}.$$

Problem 1.53. (2001)$(*)$ *Let* $a_0 = \sqrt{2}$, $b_0 = 2$ *and further* $a_{n+1} = \sqrt{2 - \sqrt{4 - a_n^2}}$ *and* $b_{n+1} = 2b_n/(2 + \sqrt{4 + b_n^2})$ *for all* $n \geqslant 0$. *Show that both* a_n *and* b_n *are decreasing and converge to zero. Further show that the sequence* $2^n a_n$ *is increasing and that* $2^n b_n$ *is decreasing, but that they converge to the same limit. Finally, show that for some* $C > 0$, *it holds that* $0 < b_n - a_n < C/8^n$ *for all* $n \in \mathbb{N}$.

Problem 1.54. (2004) *Let* $S \subset \mathbb{R}$ *be such that* $|s_1 + s_2 + \ldots + s_k| < 1$ *for any finite subset* $\{s_1, \ldots, s_k\} \subset S$. *Show that* S *is finite or countable.*

Problem 1.55. (2004) *For each* $n \geqslant 2$, *let* S_n *be the set of all sums* $\sum_{k=1}^{n} x_k$ *where we require* $0 \leqslant x_1, \ldots, x_n \leqslant \pi/2$ *and*

$$\sum_{k=1}^{n} \sin(x_k) = 1.$$

Show that S_n *is an interval. Let* l_n *be its length. Find* $\lim_{n\to\infty} l_n$.

Problem 1.56. (2010) *Let* $a_1 = \sqrt{5}$ *and let* $a_{n+1} = a_n^2 - 2$. *Determine*

$$\lim_{n\to\infty} \frac{\prod_{k=1}^{n} a_k}{a_{n+1}}.$$

Problem 1.57. (2010)($*$) *Let* $-1 \leqslant a, b, c \leqslant 1$ *be real such that* $1 + 2abc \geqslant a^2 + b^2 + c^2$. *Show that for every positive integer* n, *it holds that* $1 + 2(abc)^n \geqslant a^{2n} + b^{2n} + c^{2n}$.

Problem 1.58. (2010) *A sequence* (x_n) *of real numbers satisfies* $x_{n+1} = x_n \cos(x_n)$ *for all* $n \geqslant 1$. *Does this sequence converge for all choices of* x_1? *What about* $x_{n+1} = x_n \sin(x_n)$?

Problem 1.59. (2010) *Let* $a_0, \ldots, a_n > 0$ *be real numbers such that* $a_{k+1} - a_k \geqslant 1$ *for all* $0 \leqslant k < n$. *Show that*

$$1 + \frac{1}{a_0}\left(1 + \frac{1}{a_1 - a_0}\right) \cdots \left(1 + \frac{1}{a_n - a_0}\right) \leqslant \left(1 + \frac{1}{a_0}\right)\left(1 + \frac{1}{a_1}\right) \cdots \left(1 + \frac{1}{a_n}\right).$$

Problem 1.60. (2011) *Let* (a_n) *be a sequence of real numbers with* $1/2 < a_n < 1$ *for all* $n \geqslant 0$. *Define another sequence* x_n *by* $x_0 = a_0$ *and further*

$$x_{n+1} = \frac{a_{n+1} + x_n}{1 + a_{n+1} x_n}.$$

Does this sequence always converge? What are its possible limits?

2 Algebra I

Algebra in essence is the study of groups, rings, modules and related objects. Basic knowledge of groups is indispensible at the IMC, though often some mild ring theoretic notions are also very useful to keep in mind. We specialise in these notes to two of the main applications of the field of algebra; number theory, and the study of polynomials and the rings which describe their arithmetic. This section is aimed to secure a good basis of abstract algebra, upon which all other concepts build.

2.1 Group Theory

A group is a set endowed with an associative, binary operation with an identity element for which each element admits an inverse. If this operation is commutative, we say that the group is *abelian*. There are many ways to denote the unit element; sometimes the letter e is used, but when the group is written multiplicatively, we also often write 1. For additive notation, typical for abelian groups, 0 is most common. The following result is fundamental in the theory of groups.

Proposition 2.1. (Lagrange) *Let G be a finite group and denote $n = \#G$. Then for any $g \in G$ it holds that $g^n = 1$. Similarly, for any subgroup H, it holds that $\#G = [G : H] \cdot \#H$.*

The following result is sometimes useful.

Proposition 2.2. (Cauchy) *Let G be a finite group and let $p \mid \#G$. Then there exists some $g \in G \setminus \{1\}$ with $g^p = 1$.*

One of the most important family of groups to be familiar with is the family of *symmetric groups*, written S_n. These are the groups of all permutations of any n element set and it is of cardinality $n!$. Further, it comes with a natural sign map $S_n \to \{\pm 1\}$ that reveals some additional structure. Its kernel is denoted A_n and is the subgroup of all even permutations. A conjugacy class in a group G is defined as any of the orbits for the conjugation action of G on itself. With this definition, it is easy to prove the following.

Lemma 2.3. *The size of every conjugacy class must divide the order of the group.*

The centre $Z(G)$ of a group G is defined as the set of elements that commute with all other elements of G. Clearly, $Z(G) = G$ if and only if G is abelian.

Lemma 2.4. *Let G be a finite group of order p^k for some prime p and $k \geqslant 1$. Then the centre of G is non-trivial.*

The commutator subgroup $[G, G] \subset G$ is the subgroup generated by all expressions of the form $ghg^{-1}h^{-1}$ for $g, h \in G$. The quotient $G/[G, G]$ is always abelian and $[G, G]$ is minimal for this property. It is a fun fact to know that $[S_n, S_n] = A_n$.

2.2 Actions and subgroups

Normal subgroups N of a group G are precisely those subgroups that can be written as $N = \ker(f)$ for some group homomorphism $f : G \to H$. By definition they are stable under conjugation by any element in G.

The most important of the isomorphism theorems states that $G/N \cong \mathrm{im}(f)$. If $H \subset G$ is a subgroup, then we also know that $H/(H \cap N) \cong HN/N$. Finally, if both $N_1 \subset N_2 \subset G$ are all normal subgroups, then $(G/N_1)/(N_2/N_1) \cong G/N_2$.

In short, an action of a group G on a set X is given by a homomorphism $G \to S(X)$, where $S(X)$ denotes the set of permutations of the set X. This can be used to prove the existence of large normal subgroups given only a large subgroup.

Proposition 2.5. *Let G be a group and $H \subset G$ a subgroup with $[G : H] = n$. Then there exists a normal subgroup N of G, contained in H, such that $[G : N] \mid n!$.*

This can be used to conclude that some subgroups must be normal.

Lemma 2.6. *Let G be a finite group and $H \subset G$ a subgroup such that $\#H$ and $([G : H] - 1)!$ are coprime. Then H is normal. In particular, if $[G : H]$ is the smallest prime dividing $\#G$, then H is normal.*

If a finite group G acts on a finite set X, we let Gx for $x \in X$ denote the orbit of x and we let G_x denote the stablisiser subgroup. Then $\#Gx = [G : G_x]$, and as such,

$$\#X = \sum_{x \in \mathrm{orb}(X)} [G : G_x].$$

Theorem 2.7. (Burnside) *Let a finite group G act on a finite set X and let $\mathrm{Fix}(g)$ for $g \in G$ denote the number of fixed points for the action of g on X. If $\#X/G$ denotes the number of orbits, we have*

$$\#X/G = \frac{1}{\#G} \sum_{g \in G} \#\mathrm{Fix}(g).$$

It is sometimes useful to consider the Sylow subgroups of a finite group.

Theorem 2.8. (Sylow) *Let G be a finite group and let p be a prime such that $p^k \mid \#G$ for some $k \geqslant 1$. Then G contains a subgroup of size p^k. If k is chosen maximally, we call any such subgroup a p-Sylow subgroup of G. Any two such subgroups are conjugate in G. If s_p denotes the number of p-Sylow subgroups, then $s_p \equiv 1 \mod p$ and $s_p \mid \#G$. Finally, a p-Sylow subgroup is normal if and only if it is the only one.*

2.3 Ring Theory

A ring is an abelian group with a distributive, associative multiplication structure. Often it is also assumed to have a neutral element 1 for this multiplication. Rings contain a number of special elements.

- We say some $a \in R$ is a *unit* if for some $b \in R$, it holds that $ab = ba = 1$. The subset of units in R is typically denotes $R^\times$ and this forms a group. Note that elements in general can have a left-inverse, but not a right-inverse, and the other way around.
- Some $a \in R$ is a *zero divisor* if for some $b \in R$, it holds that $ab = 0$ or $ba = 0$.
- We say some $a \in R$ is an idempotent if $a^2 = a$.
- We say some $a \in R$ is nilpotent if $a^n = 0$ for some $n \geqslant 1$.

From now on we will assume that all rings are commutative. If R contains no zero divisors, we say it is a *domain*. If $R^\times = R \setminus \{0\}$, we say that R is a *field*.

An *ideal* $I \subset R$ is an R-submodule of R; in other words, I is an abelian group and $ri \in I$ for all $r \in R$ and $i \in I$. Ideals are precisely the possible kernels of ring morphisms. We say an ideal I is *prime* if $ab \in I$ means that either $a \in I$ or $b \in I$. An ideal is called *maximal* if no proper ideal of R contains it.

Lemma 2.9. *An ideal $I \subset R$ is prime if and only if R/I is a domain. Similarly, I is maximal if and only if R/I is a field.*

We say an ideal is *principal* if it is of the form aR for some $a \in R$. If every ideal in a domain R is of that form, we say that R is a *principal ideal domain*. We have the following result about the structure of modules over such rings.

Theorem 2.10. *Let R be a principal ideal domain and M a finitely generated R-module. Then there exist an integer n and non-zero ideals $I_1, \ldots, I_k$ of R such that*

$$M \cong R^n \oplus R/I_1 \oplus \cdots \oplus R/I_k.$$

Because $\mathbb{Z}$ is a principal ideal domain and abelian groups are just $\mathbb{Z}$-modules, this implies the following.

Corollary 2.11. *Let A be a finite abelian group. Then there exist prime numbers $p_1, \ldots, p_k$ and positive integers $e_1, \ldots, e_k$ such that*

$$A \cong \mathbb{Z}/p_1^{e_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{e_k}\mathbb{Z}.$$

2.4 Examples

Example 2.12. Let S be a set with a binary operation $*$ such that $x * (y * x) = y$ for all $x, y \in S$. Prove that for any $a, b \in S$, the equations $a * x = b$ and $x * a = b$ both have a unique solution in S.

Solution: We claim that $x = b * a$ is the only solution to the equation $a * x = b$. Indeed, this value of x works, as $a * (b * a) = b$ by assumption. Now let $x \in S$ be any solution to the equation $a * x = b$. Then it follows that also $a = x * (a * x) = x * b$. In turn, we may then also deduce that $b * a = b * (x * b) = x$. We leave the other equation to the reader. △

Example 2.13. *(IMC 2018)* Does there exist a field such that its multiplicative group is isomorphic to its additive group?

Solution: Let k denote such a field and let $f : k^\times \to k$ be an isomorphism. Now note that

$$2f(-1) = f((-1)^2) = f(1) = 0.$$

As $f(-1) \neq 0$ and k is a field, this implies that $2 = 0$ and as such, the field k is of characteristic 2. But then for any $x \in k^\times$, we have

$$f(x^2) = 2f(x) = 0 = f(1),$$

and as such, $x^2 = 1$ for all $x \in k$. But again as k is a field, this implies that $x = \pm 1$. However, now k is either $\mathbb{F}_2$ or $\mathbb{F}_3$, but for cardinality reasons, these fields do not admit such an isomorphism f. So, such a field does not exist. △

2.5 Exercises

Problem 2.1. *Let $*$ denote an operation on a set G that satisfies the left-axioms of a group; so $*$ is associative, there is a left-unit $e \in G$ such that $e * g = g$ for all $g \in G$ and for each $g \in G$, there is a left-inverse g_L such that $g_L * g = e$. Prove that g_L is also a right-inverse of g and show that e is also a right-unit.*

Problem 2.2. *Let S be a set with a binary operation $*$. Suppose that for any $a, b \in S$, it holds that $(a * b) * a = b$. Prove that then also $a * (b * a) = b$ for all $a, b \in S$.*

Problem 2.3. $(*)$ *Let $S \subset \mathbb{R}$ be a set that is closed under multiplication. We now write $S = T \cup U$ with $T \cap U = \varnothing$. Suppose that for any $a, b, c \in T$ it holds that $abc \in T$ and that for any $a, b, c \in U$ it holds that $abc \in U$. Prove that at least one of T and U is also closed under multiplication.*

Problem 2.4. *Let $*$ denote a binary operation on a set S such that for any $x, y \in S$, it holds that $x * (x * y) = y$ and $(y * x) * x = y$. Prove that $*$ is commutative. Is it also necessarily associative?*

Problem 2.5. $(*)$ *Let* S *be a set and let* $*$ *be a binary operation on* S *with* $x * x = x$ *and* $(x * y) * z = (y * z) * x$ *for all* $x, y, z \in S$*. Prove that* $*$ *is both commutative and associative.*

Problem 2.6. $(*)$ *Let* $*$ *be an associative binary operation on a set* S *such that* $a * b = b * a$ *if and only if* $a = b$*. Prove that* $a * (b * c) = a * c$ *for any* $a, b, c \in S$*. Can you give an example of such an operation?*

Problem 2.7. $(*)$ *Let* $*$ *be a binary operation on* $\mathbb{R}$ *such that* $(a * b) * c = a + b + c$ *for any* $a, b, c \in \mathbb{R}$*. Prove that* $a * b = a + b$ *for any* $a, b \in \mathbb{R}$*.*

Problem 2.8. *Let* $*$ *and* $\star$ *be two binary operations on a set* S *with respective unit elements* e *and* f*, satisfying*

$$(x \star y) * (u \star v) = (x * u) \star (y * v)$$

for any $x, y, u, v \in S$*. Prove that* $e = f$*, that* $* = \star$ *and that both operations are commutative.*

Problem 2.9. *Let* $*$ *be a binary operation on a set* S *such that there is some* $e \in S$ *with the property that* $x * e = x$ *for all* $x \in S$*, further satisfying that* $(x * y) * z = (z * x) * y$ *for all* $x, y, z \in S$*. Prove that* $*$ *is both associative and commutative.*

Problem 2.10. $(*)$ *Let* $*$ *be a binary operation on* $\mathbb{Q}$ *that is both associative and commutative, further satisfying* $0 * 0 = 0$ *and* $(a + c) * (b + c) = a * b + c$ *for any* $a, b, c \in \mathbb{Q}$*. Prove that either* $a * b = \max(a, b)$ *for all* $a, b \in \mathbb{Q}$*, or* $a * b = \min(a, b)$ *for all* $a, b \in \mathbb{Q}$*.*

2.6 IMC Problems

Problem 2.11. (1996) *Let* G *be the subgroup of* $GL_2(\mathbb{R})$ *generated by the matrices*

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let H *consist of those matrices in* G *for which both diagonal entries are equal to 1. Prove that* H *is an abelian subgroup of* G*, but that* H *as a group is not finitely generated.*

Problem 2.12. (1998) *Prove that for* $n \in \{3, 5\}$ *and any permutation* $\pi_1 \in S_n$*, there exists some* $\pi_2 \in S_n$ *such that* π_1 *and* π_2 *together generate all of* S_n*. Is this statement true for* S_4*?*

Problem 2.13. (1999) *Suppose that in a not necessarily commutative ring* R *without 1, the square of every element is 0. Prove that* $abc + abc = 0$ *for all* $a, b, c \in R$*.*

Problem 2.14. (2000) *Let* R *be not necessarily commutative ring of characteristic zero. Let* e*,* f *and* g *be idempotent elements of* R *satisfying* $e + f + g = 0$*. Prove that* $e = f = g = 0$*.*

Problem 2.15. (2001) *Let* r, s, t *be pairwise coprime integers. Let* G *be an abelian group and* $a, b \in G$ *such that* $a^r = b^s = (ab)^t = 1$*. Prove that* $a = b = 1$*. Does this conclusion also hold if* G *is not necessarily abelian?*

Problem 2.16. (2003) *Let* $a_1, \dots, a_{51}$ *be non-zero elements of a field. We simultaneuously replace each element with the sum of the remaining 50 ones, obtaining a new sequence* $b_1, \dots, b_{51}$. *If this new sequence is a permutation of the original, what can be the characteristic of the field?*

Problem 2.17. (2005)($*$) *For any group* G *and* $m \in \mathbb{N}$, *let* $G(m) \subset G$ *be the subgroup generated by the* m*th powers of elements of* G. *Suppose that* $G(m)$ *and* $G(n)$ *are commutative. Prove that* $G(\gcd(m, n))$ *is also commutative.*

Problem 2.18. (2005)($*$) *Prove that for any* $r \in \mathbb{Q}(\sqrt{7})$, *there exists a matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \setminus \{\pm 1\}$ *such that* $ar + b = r(cr + d)$.

Problem 2.19. (2007) *Let* G *be a finite group. For arbitrary subsets* $U, V, W \subset G$, *denote by* N_{UVW} *the number of triples* $(x, y, z) \in U \times V \times W$ *for which* $xyz = 1$. *Suppose that* G *is partitioned into three sets* A, B *and* C. *Prove that* $N_{ABC} = N_{CBA}$.

Problem 2.20. (2008) *Does there exist a finite group* G *with a normal subgroup* H *such that* $\#\mathrm{Aut}(H) > \#\mathrm{Aut}(G)$*?*

Problem 2.21. (2010) *Let* G *be a subset of the symmetric group* S_n *such that for every* $\pi \in G \setminus \{1\}$, *there exists a unique* k *for which* $\pi(k) = k$. *Show that this* k *is the same for all* $\pi \in G \setminus \{1\}$.

Problem 2.22. (2012) *Given an integer* $n > 1$, *two players, A and B, play the following game. Taking turns, they select one element from the symmetric group* S_n, *with the rule that it is forbidden to select an element that has been previously selected. The same ends when the selected elements generate all of* S_n. *The player who moves last, loses the game. If A starts, which player has a winning strategy?*

Problem 2.23. (2012)($*$) *Let* $c \geqslant 1$ *be a real number. Let* G *be an abelian group and let* $A \subset G$ *be a finite set satisfying* $|2 * A| \leqslant c|A|$, *where for any positive integer* k, *the set* $(k+1) * A := \{x + y \mid x \in k * A, y \in A\}$ *is defined inductively. Prove that* $|k * A| \leqslant c^k |A|$ *for any* $k \in \mathbb{N}$.

Problem 2.24. (2016)($*$) *For every permutation* $\pi \in S_n$, *we let* $\mathrm{inv}(\pi)$ *denote the number of pairs* $1 \leqslant i < j \leqslant n$ *with* $\pi(i) > \pi(j)$. *Let* $f(n)$ *denote the number of* $\pi \in S_n$ *such that* $\mathrm{inv}(\pi)$ *is divisible by* $n + 1$. *Prove that there exist infinitely many primes such that* $p \cdot f(p-1) > (p-1)!$ *and infinitely many primes for which* $p \cdot f(p-1) < (p-1)!$.

Problem 2.25. (2016) *Let* n *be a positive integer and suppose that there exists a function* $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ *satisfying the properties that* $f(x) \neq x$ *but* $f(f(x)) = x$ *and further* $f(f(f(x+1)+1)+1) = x$ *for all* $x \in \mathbb{Z}/n\mathbb{Z}$. *Prove that* $n \equiv 2 \mod 4$.

Problem 2.26. (2020) *Let* G *be a group and* $n \geqslant 2$ *an integer. Let* H_1 *and* H_2 *be subgroups of* G *that satisfy* $[G : H_1] = [G : H_2] = n$ *and* $[G : (H_1 \cap H_2)] = n(n-1)$. *Prove that* H_1 *and* H_2 *are conjugate in* G.

Problem 2.27. (2021) *For a prime number* p, *show that there is no injective group homomorphism* $GL_2(\mathbb{Z}/p\mathbb{Z}) \to S_p$.

3 Linear Algebra I

Even very basic notions in linear algebra can make for interesting and challenging problems. Every year at the IMC, there will be at least one problem about this seemingly completely understood field of mathematics, and often there are even more. It is therefore imperative to familiarise yourself with the following concepts and tricks, which are most often exploited at the IMC.

3.1 Rank and nullity

We start by setting some notation, which can be found in any first year's course on the matter. Let k be a field and let V and W be finite-dimensional k-vector spaces. With very few exceptions, we will work with $k \in \{\mathbb{R}, \mathbb{C}\}$. Let $A : V \to W$ be a linear map.

Definition 3.1. The dimension of the image of A is called the *rank* of A, denoted $\mathrm{rk}(A)$. The dimension of the kernel of A is called the *nullity* of A, denoted $\mathrm{nul}(A)$.

These two quantities satisfy a few very basic properties, which we leave to the reader to convince or remind themselves of.

Proposition 3.2. *Let* $A : V \to W$ *and* $B : W \to U$ *be linear maps. Then:*

- *We have* $\mathrm{rk}(A) \leqslant \min\big(\dim_k(V), \dim_k(W)\big)$.
- *We have* $\mathrm{rk}(BA) \leqslant \min\big(\mathrm{rk}(A), \mathrm{rk}(B)\big)$.
- *If* $\mathrm{rk}(A) = \dim_k(W)$, *then* $\mathrm{rk}(BA) = \mathrm{rk}(B)$.
- *If* $\mathrm{rk}(B) = \dim_k(W)$, *then* $\mathrm{rk}(BA) = \mathrm{rk}(A)$.
- *We have* $\mathrm{rk}(A + B) \leqslant \mathrm{rk}(A) + \mathrm{rk}(B)$.
- *We have* $\mathrm{rk}(A) + \mathrm{nul}(A) = \dim_k(V)$.
- *We have* $\mathrm{rk}(A) + \mathrm{rk}(B) \leqslant \mathrm{rk}(BA) + \dim_k(W)$. (Sylvester's Inequality)

If we choose bases for both the source and the target of a linear map A, we may represent A as a matrix. In this language, the rank of A is equal to the greatest number of linearly independent columns we can find, which is the same as the greatest number of linearly independent rows that we can find. This number does not change when adding other columns some number of times to a given column, and similarly for rows. IMC problems are most often phrased in terms of matrices, but it can help to think about matrices in the language of linear operators instead, not in small part because many of the above results become much more intuitive in this context.

3.2 Similarity and commuting

We continue with the following cute trick that seems innocent, but at times proves to be supremely useful. Namely, matrices do not commute in general, but in rare occassions we are allowed to swap matrices around.

Lemma 3.3. *Let* A *and* B *be* $n \times n$*-matrices. Then*

$$AB = I_n \iff BA = I_n.$$

We also record the following fact about commuting matrices.

Proposition 3.4. *Let* A *and* B *be* $n \times n$*-matrices that commute; i.e. that satisfy* $AB = BA$*. Then they share a common eigenvector.*

The following theorem is the focus point of many first courses on linear algebra.

Theorem 3.5. *Every real symmetric matrix is diagonalisable. In fact, there exists an orthogonal basis of eigenvectors, and all eigenvalues are real.*

The concept of being diagonalisable is closely related to that of similar matrices; we say A and B are similar if for some invertible matrix P, we have $B = PAP^{-1}$. Similar matrices represent the same linear map, just for a different choice of basis. As such, almost all meaningful properties of similar matrices coincide, from trace and determinant to eigenvalues and rank. If v is an eigenvector for A, then Pv is one for PAP^{-1}. The following result is even more precise.

Proposition 3.6. *The following equality of characteristic polynomials is true:* $p_A = p_{PAP^{-1}}$*. In particular, we have* $p_{AB} = p_{BA}$*.*

Not every matrix is diagonalisable, but we can always get close.

Lemma 3.7. (Schur) *For any square matrix* A*, there exists some (unitary) matrix* Q *such that* QAQ^{-1} *is upper-triangular.*

This can be strengthened; even if a matrix A is not diagonalisable, it can always be brought into Jordan normal form. This is a special kind of upper-triangular matrix, which consists of blocks of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix}$$

placed on the diagonal, with zeroes elsewhere; here the λ are the eigenvalues of A. Finally, sometimes when tasked to prove something for general matrices A, we may reduce to the case of diagonalisable matrices for free, which can make a huge difference.

Namely, if the problem asks to prove a polynomial identity about all or a certain class of matrices, then often the diagonalisable matrices are dense inside this set. If a polynomial identity holds on a dense subset, if must hold on the full space. This can yield quicker proofs of many known theorems and is sometimes useful at the IMC. This can be used to prove the non-obvious but sometimes useful fact that any matrix A is similar to its transpose A^T. Even though some linear algebra questions, even at the IMC, are direct applications of the material discussed in this brief introduction, most will require a thorough understanding of the interplay between linear maps and matrices, and will require one or multiple creative and key insights to solve.

3.3 Examples

Example 3.8. Let A be a real $n \times n$-matrix satisfying $A^3 = 0$. Let $\mathbb{1}$ be the $n \times n$ matrix containing only 1's. For which n can we conclude that $\det(A + \mathbb{1}) = 0$?

Solution: We use Sylvester's inequality to deduce that

$$2\text{rk}(A) \leqslant \text{rk}(A^2) + n.$$

Similarly, we may apply it again to find that

$$\text{rk}(A) + \text{rk}(A^2) \leqslant \text{rk}(A^3) + n = n.$$

Adding these inequalities gives us

$$3\text{rk}(A) + \text{rk}(A^2) \leqslant \text{rk}(A^2) + 2n \implies \text{rk}(A) \leqslant 2n/3.$$

Now clearly $\text{rk}(\mathbb{1}) = 1$, so we find that

$$\text{rk}(A + \mathbb{1}) \leqslant \text{rk}(A) + \text{rk}(\mathbb{1}) \leqslant 2n/3 + 1 < n$$

as soon as $n > 3$. If a matrix does not have full rank, it cannot be invertible; as such, $\det(A + \mathbb{1}) = 0$ as soon as $n > 3$. To see that the claim fails for $n < 3$, we invite the reader to consider the matrices

$$A = \begin{pmatrix} 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

respectively. This solves the problem. $\triangle$

Example 3.9. *(IMC 2003)* Let A and B be $n \times n$-matrices satisfying the equation $AB + A + B = 0$. Prove that A and B commute and that $rk(A) = rk(B)$.

Solution: Note that

$$AB + A + B = 0 \iff AB + A + B + I = I \iff (A + I)(B + I) = I.$$

This means that A and B are each other's inverses, and as such, we also have that

$$(B+I)(A+I) = I \iff BA + A + B + I = I \iff BA + A + B = 0.$$

Comparing this result to the given equation yields that $AB = BA$, as desired. For the statement about ranks, note that

$$A(B+I) = AB + A = -B.$$

Since A and B now differ up to an invertible matrix, their ranks must be the same. $\triangle$

3.4 Exercises

Problem 3.1. *Let $n > k$ be two positive integers and let A_i for $i = 1, \dots, k$ be an $n \times n$-matrix with $\mathrm{rk}(A_i) = n - 1$. Prove that $A_1 \cdots A_k \neq 0$.*

Problem 3.2. *Let A and B be two nilpotent matrices of the same size. Show that if A and B commute, then $A + B$ is also nilpotent. Is this still true if A and B do not commute?*

Problem 3.3. *Let A be a matrix satisfying $A^2 = I_n$. Show that $\mathrm{rk}(A + I) + \mathrm{rk}(A - I) = n$. Does the converse hold?*

Problem 3.4. *Determine all invertible matrices A with non-negative real entries such that A^{-1} also has only non-negative real entries.*

Problem 3.5. *Determine the rank of the $n \times n$-matrix A with entries $a_{ij} = (i + j)^2$.*

Problem 3.6. *Let A and B be two* distinct *matrices satisfying $A^3 = B^3$ and $A^2B = B^2A$. Can $A^2 + B^2$ be invertible?*

Problem 3.7. *Let P and Q be square matrices of the same size, satisfying $P^2 + P = Q^2 + Q$ and such that $P + Q + I$ is invertible. Show that $\mathrm{rk}(P) = \mathrm{rk}(Q)$.*

Problem 3.8. *Let A be an $n \times n$-matrix with $a_{ij} \in \{0, 1\}$ for all $1 \leqslant i, j \leqslant n$. Suppose further that $a_{ii} = 0$ and $a_{ij} + a_{ji} = 1$ for all i, j. Prove that $\mathrm{rk}(A) \geqslant n - 1$.*

Problem 3.9. *Let A be a 3×3-matrix with rational entries. Suppose that $A^8 = I_3$. Prove that even $A^4 = I_3$.*

Problem 3.10. *Let A be a 2×2-matrix with integer coefficients. Suppose that $A^n = I_2$ for some n coprime to 6. Prove that $A = I_2$.*

Problem 3.11. *Let A and B be complex 2×2-matrices such that $AB - BA = B^2$. Prove that A and B commute.*

Problem 3.12. *Let A and B be complex 2×2-matrices such that $A^2 + B^2 = 2AB$. Prove that A and B commute and that their traces are equal.*

Problem 3.13. *Let* A, B, C *be* $n \times n$ *matrices such that* C *commutes with both* A *and* B*. Suppose further that* $C^2 = I_n$ *and* $AB = 2(A + B)C$*. Show that* A *and* B *also commute. If further* $A + B + C = 0$*, show that* $\mathrm{rk}(A - C) + \mathrm{rk}(B - C) = n$*.*

Problem 3.14. $(*)$ *Let* A *be a* 3×2*-matrix and* B *a* 2×3*-matrix. Suppose that*

$$AB = \begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}.$$

Determine BA*.*

Problem 3.15. *Let* M *be a* 3×3 *magic square and view it as a matrix. Prove that for any odd positive integer, the matrix* M^n *is also a magic square.*

Problem 3.16. $(*)$ *Let* S *be the set of* 2×2*-matrixes of the form*

$$\begin{pmatrix} a & a + d \\ a + 2d & a + 3d \end{pmatrix}.$$

Find all $M \in S$ *such that for some* $k > 1$*, we have that also* $M^k \in S$*.*

3.5 IMC Problems

Problem 3.17. (1994) *Let* $n \geqslant 2$ *be an integer and let* A *be an invertible* $n \times n$*-matrix with positive real entries. Show that* A^{-1} *contains at most* $n^2 - 2n$ *zeroes. How many zeroes occur in the inverse of*

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & 2 & \dots & 2 \\ 1 & 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & 2 & \dots & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 1 & 2 & \dots & \dots \end{pmatrix}?$$

Problem 3.18. (1994) *Let* A *be a diagonal matrix and let* $d(\lambda)$ *for* $\lambda \in \mathbb{R}$ *denote the number of times* λ *occurs on the diagonal of* A*. Show that the dimension of the space of matrices* B *that commute with* A *is given by* $\sum d(\lambda)^2$*.*

Problem 3.19. (1995) *Let* X *be an invertible* $n \times n$*-matrix with columns* $X_1, X_2, \dots, X_n$*. Let* Y *be the matrix with columns* $X_2, X_3, \dots, X_n, 0$*. Show that the matrix* $A = YX^{-1}$ *and the matrix* $B = X^{-1}Y$ *both have rank* $n - 1$*, yet are nilpotent.*

Problem 3.20. (1995) *Let* A *and* B *be real* $n \times n$*-matrices with the property that there exist* $n + 1$ *distinct real numbers* $t_0, \dots, t_n$ *such that the matrices* $A + t_i B$ *are nilpotent for all* $0 \leqslant i \leqslant n$*. Show that both* A *and* B *are nilpotent themselves.*

Problem 3.21. (1995) *Let A be a real $n \times n$-matrix with the property that any $u \in \mathbb{R}^n$ is orthogonal to Au. Show that A is skew-symmetric.*

Problem 3.22. (1996) *Given a positive integer n, find the largest possible integer k such that there exists a set of pairwise commuting $n \times n$-matrices $A_1, \dots, A_k$ further satisfying that $A_i^2 = I_n$ for all $1 \leqslant i \leqslant k$.*

Problem 3.23. (1997) *Let A and B be real $n \times n$-matrices such that $A^2 + B^2 = AB$. Prove that if $BA - AB$ is invertible, then n is divisible by 3.*

Problem 3.24. (1998) *Let $V = \mathbb{R}^{10}$ and let $U_1 \subset U_2 \subset V$ be subspaces with $\dim(U_1) = 3$ and $\dim(U_2) = 6$. Let $\mathcal{E}$ be the space of linear maps $T : V \to V$ such that $T(U_1) \subset U_1$ and $T(U_2) \subset U_2$. Determine $\dim(\mathcal{E})$.*

Problem 3.25. (1998) *Let V be a real vector space and let $f, f_1, \dots, f_k : V \to \mathbb{R}$ be linear maps. Suppose that $f(x) = 0$ whenever $f_1(x) = \dots = f_k(x) = 0$. Show that f can be written as a linear combination of the $f_1, \dots, f_k$.*

Problem 3.26. (2002) *Let A be a complex $n \times n$-matrix. Show that $A\overline{A} = I_n$ if and only if for some invertible matrix S, we have $A = S\overline{S}^{-1}$.*

Problem 3.27. (2004) *Let A be a real 4×2-matrix and B a real 2×4 matrix such that*

$$AB = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

Determine BA.

Problem 3.28. (2005) *Let $A = (a_{ij})$ be the $n \times n$-matrix given by $a_{ij} = i + j$. Find $\operatorname{rk}(A)$.*

Problem 3.29. (2006)(∗) *Let $v_0 = 0$ and let $v_1, \dots, v_{n+1} \in \mathbb{R}^n$ be such that the norm $|v_i - v_j|$ is rational for all $0 \leqslant i, j \leqslant n+1$. Prove that $v_1, \dots, v_{n+1}$ are linearly dependent over $\mathbb{Q}$.*

Problem 3.30. (2007) *Let $n \geqslant 2$ be an integer. What is the smallest possible rank of an $n \times n$-matrix whose coefficients are precisely the numbers $1, 2, \dots, n^2$? What about the greatest possible rank?*

Problem 3.31. (2008) *For each positive integer k, find the smallest positive integer n for which there exists real $n \times n$-matrices $A_1, \dots A_k$ such that $A_i^2 = 0$ and $A_iA_j = A_jA_i$ for all indices $1 \leqslant i, j \leqslant k$, with the additional property that $A_1 \cdots A_k \neq 0$.*

Problem 3.32. (2009) *Let A, B and C be $n \times n$-matrices with A invertible. Suppose that the equation $C(A - B) = A^{-1}B$ holds. Prove that $(A - B)C = BA^{-1}$.*

Problem 3.33. (2010)(∗) *Let A be a symmetric matrix over $\mathbb{F}_2$ all of whose diagonal entries are zero. Prove that for each positive integer n, each column of A^n contains at least one zero.*

Problem 3.34. (2012) *Let n be a positive integer. Determine the smallest possible rank of an $n \times n$-matrix with only zeros on its diagonal and positive entries elsewhere.*

Problem 3.35. (2013)($*$) *Let $v_1, \dots, v_n$ be unit vectors in $\mathbb{R}^n$. Show that there exists some unit vector u such that $|\langle u, v_i \rangle| \leqslant 1/\sqrt{n}$ for all $1 \leqslant i \leqslant n$.*

Problem 3.36. (2016) *Let n be a positive integer. We say a set of $n \times n$-matrices $\{A_1, \dots, A_k\}$ is* interesting *if $A_i^2 \neq 0$ for all $1 \leqslant i \leqslant k$ and $A_i A_j = 0$ for all $i \neq j$. Show that the largest possible interesting set is precisely of size n.*

Problem 3.37. (2018) *Determine all rational numbers a for which the matrix*

$$\begin{pmatrix} a & -a & -1 & 0 \\ a & -a & 0 & -1 \\ 1 & 0 & a & -a \\ 0 & 1 & a & -a \end{pmatrix}$$

is the square of a matrix with all rational entries.

Problem 3.38. (2018) *Let k be a positive integer. Determine the smallest integer n with the property that there exist k non-zero vectors $v_1, \dots v_k \in \mathbb{R}^n$ such that for any two indices i, j with $|i-j| > 1$, the vectors v_i and v_j must be orthogonal. How does the answer change if we require v_i and v_j to be orthogonal* if and only if *$|i-j| > 1$?*

Problem 3.39. (2019) *Determine all positive integers n for which there exist $n \times n$ real invertible matrices A and B satisfying $AB - BA = B^2 A$.*

Problem 3.40. (2021) *Let A be a real $n \times n$-matrix such that $A^3 = 0$. Determine all matrices X such that $X + AX + XA^2 = A$.*

Problem 3.41. (2021) *Let p be a prime number. Show that there is no injective group homomorphism from $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ to the permutation group S_p.*

Problem 3.42. (2022) *Let $A_1, \dots, A_k$ be $n \times n$-matrices satisfying $A_i^2 = A_i$ and further $A_i A_j = -A_j A_i$ for all $1 \leqslant i < j \leqslant k$. Prove that for some i, we must have $\mathrm{rk}(A_i) \leqslant n/k$.*

Problem 3.43. (2022) *Let A, B and C be $n \times n$ matrices with complex entries satisfying*

$$A^2 = B^2 = C^2 \quad \textit{and} \quad B^3 = ABC + 2I.$$

Prove that $A^6 = I$.

4 Analysis II

Before moving on to the theory of functions, continuity and integration, we will momentarily pause to consider a natural extension of the realm of sequences: namely, the concept of series. Problems about series are ubiquitous at the IMC and as such deserve their own section for practice.

4.1 Series

For a sequence (a_n) of real numbers, the sum $\sum_{n=1}^{\infty} a_n$ is the *series* associated with a_n. If this series converges, then $\lim_{n\to\infty} a_n = 0$, but the converse is not true. However, if (a_n) does not converge to zero, then the series cannot converge either. Some power series are particularly important to remember, as they can be used in myriad ways. Well known is

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k; \quad \text{in particular,} \quad \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k.$$

Integrating the latter yields

$$\log(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}; \quad \text{inverting yields} \quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Of course one may only use these power series on their domains of convergence, which is $|x| < 1$ for the former and all of $\mathbb{R}$ for the latter. Similarly, the series

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad \text{and} \quad \cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

converge on all of $\mathbb{R}$. To determine if a given series converges, it is often good practice to compare it to a series of which convergence or divergence is already known. For example, it is useful to know that

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \quad \text{converges precisely when } \alpha > 1.$$

The following theorem is sometimes useful to settle edge-cases when it comes to convergence.

Proposition 4.1. (Abel's Theorem) *Suppose that a formal power series converges on an open interval $(-r, r)$ to a function f. If the power series at $x = r$ also converges to some value L, then*

$$L = \lim_{x\to r} f(x).$$

The following proposition records two more criteria that are sometimes useful.

Proposition 4.2. *Let (a_n) be a sequence such that* $\limsup |a_{n+1}/a_n| < 1$*. Then* $\sum_{n=1}^{\infty} a_n$ *converges. Similarly, if* $\limsup |a_n|^{1/n} < 1$*, the series converges.*

These may seem complicated, but in reality these are both direct consequences of the fact that a geometric series converges precisely when the multiplicative factor is in absolute value smaller than 1.

4.2 Manipulating Series

We record here some conditions on when we can change the order of summation, which is sometimes useful but always requires some justification.

Proposition 4.3. *Let (a_n) be a sequence of real numbers. If*

$$\sum_{n=1}^{\infty} |a_n|$$

converges, so will

$$\sum_{n=1}^{\infty} a_n$$

and this limit is independent from the order of the terms.

Proposition 4.4. *Let $(a_{m,n})$ be a multi-indexed sequence of real numbers. If*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{m,n}|$$

converges, then we have that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{m,n}.$$

As in the previous section, some sequences can be written as $a_n = f(n)$ for some obvious choice of function $f : \mathbb{R} \to \mathbb{R}$. If f is positive yet decreasing, we have the following estimates:

$$\sum_{n=a+1}^{b} f(n) \leqslant \int_a^b f(x)dx \leqslant \sum_{n=a}^{b-1} f(n) \quad \text{for all } a, b \in \mathbb{N}.$$

This often allows one to get a very sharp bound on the growth of a series of this kind; even though it does not give much information about the precise value of the series, it is certainly true that

$$\sum_{n=1}^{\infty} f(n) < \infty \iff \int_1^{\infty} f(x)dx < \infty,$$

which is often quite useful for establishing convergence or divergence of a series, provided computing the integral is much easier than analysing the series directly. When dealing with infinite products, the following proposition is often critical.

Proposition 4.5. *Let* (a_n) *be a sequence of positive numbers. Then* $L := \sum_{i=1}^{\infty} a_n < \infty$ *if and only if* $\prod_{i=1}^{\infty}(1 + a_n) < \infty$*. More precisely, if this holds, then*

$$1 + L < \prod_{i=1}^{\infty}(1 + a_n) < e^L.$$

4.3 Examples

Example 4.6. Determine

$$\sum_{n=1}^{\infty} \frac{1}{n2^n}.$$

Solution: We integrate the generating function:

$$\sum_{n=0}^{\infty} X^n = \frac{1}{1-X} \implies \sum_{n=0}^{\infty} \frac{X^{n+1}}{n+1} = \log(1-X) \quad \text{for } |X| < 1.$$

Now plug in $X = 1/2$ to find that the sequence evaluates precisely to $\log(2)$. △

Example 4.7. Prove that for any $n \in \mathbb{N}$, it holds that

$$\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{2^2}\right) \cdots \left(1 + \frac{1}{2^n}\right) < \frac{5}{2}.$$

Solution: Naively applying the proposition above, using that $1/2 + 1/4 + \ldots = 1$, we would bound this infinite product by $e > 5/2$, which is too coarse. To refine our approach, we may multiply both sides by $2/3$ to reduce to showing that the same product without the first factor is bounded above by $5/3$. Now since $1/4 + 1/8 + \ldots = 1/2$, the above proposition bounds this infinite product by $\sqrt{e} < 5/3$, completing the proof. △

4.4 Exercises

Problem 4.1. *Determine*
$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)}.$$

Problem 4.2. *Determine the sum*

$$\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} + \dots$$

Problem 4.3. *Let* (a_n) *satisfy* $a_n > 0$ *and* $a_n < a_{2n} + a_{2n+1}$ *for all* $n \geqslant 1$. *Show that* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Problem 4.4. *Let* (a_n) *be a decreasing sequence of positive real numbers. Prove that* $\sum_{n=1}^{\infty} a_n$ *converges if and only if* $\sum_{n=1}^{\infty} 2^n a_{2^n}$ *converges.*

Problem 4.5. *Determine*
$$\frac{0}{1!} + \frac{1}{2!} + \frac{2}{3!} + \dots$$

Problem 4.6. *Let* $0 < x < 1$. *Compute the product*

$$\prod_{n=0}^{\infty} (1 + x^{2^n}).$$

Problem 4.7. *For each positive integer* n, *define*

$$S_n = \sum_{\substack{x,y \leqslant n \\ x+y \geqslant n}} \frac{1}{xy}.$$

Determine $\lim_{n\to\infty} S_n$.

Problem 4.8. *Compute*
$$\frac{3}{1\cdot 2} - \frac{5}{2\cdot 3} + \frac{7}{3\cdot 4} - \frac{9}{4\cdot 5} + \dots$$

Problem 4.9. *Let the Fibonacci sequence be given by* $F_0 = F_1 = 1$ *and* $F_{n+2} = F_{n+1} + F_n$ *for all* $n \geqslant 0$. *Determine*
$$\sum_{n=1}^{\infty} \frac{F_n}{F_{n-1}F_{n+1}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{F_{n-1}F_{n+1}}.$$

Problem 4.10. *Compute the infinite product*

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$

Problem 4.11. *Prove the identity*

$$4\sum_{k=1}^{n} k\sin\left(\frac{k\pi}{2n}\right)^2 = (n+1)^2 + \cot\left(\frac{\pi}{2n}\right)^2.$$

Problem 4.12. *Determine all real* $\alpha > 0$ *with the property that*

$$\sum_{n=1}^{\infty} \frac{2012}{(n+\alpha)(16n+2012)} = 1.$$

Problem 4.13. *Let* $x_1, \dots, x_n \in \mathbb{R}$*. Show that*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \min(i,j)x_i x_j \geqslant 0.$$

Problem 4.14. *Compute*

$$\sum_{n=0}^{\infty} \frac{n}{n^4+n^2+1}.$$

Problem 4.15. *Let* $|x| < 1$ *be a real number. Determine*

$$\frac{x}{x+1} + \frac{x^2}{(x+1)(x^2+1)} + \frac{x^4}{(x+1)(x^2+1)(x^4+1)} + \dots$$

Problem 4.16. *Determine*

$$\sum_{n=1}^{\infty} \frac{1}{n(2n+1)}.$$

Problem 4.17. *Compute the series*

$$\sum_{n=1}^{\infty} \frac{1}{n(9n^2-1)}.$$

Problem 4.18. *Compute*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1}.$$

Problem 4.19. *Let* $B(n)$ *denote the number of ones in the binary expansion of the positive integer* n*. Determine*

$$\sum_{n=1}^{\infty} \frac{B(n)}{n(n+1)}.$$

Problem 4.20. *Show that the sum*

$$\sum_{n=1}^{\infty} (-1)^n \frac{\log(n)}{n^\alpha}$$

converges for all $\alpha > 0$*.*

Problem 4.21. $(*)$ *Determine the sum*

$$\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1}-2^{k+1})(3^k-2^k)}.$$

Problem 4.22. $(*)$ *Let* T *be the set of triples* (a,b,c) *of positive integers for which there exists a triangle with side lengths* a, b *and* c. *Determine*

$$\sum_{(a,b,c)\in T} \frac{2^a}{3^b 5^c}.$$

Problem 4.23. $(*)$ *Determine the double sum*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{m^2n+mn^2+2mn}.$$

Problem 4.24. $(*)$ *Find the value of the double sum*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{m^2n}{3^m(n3^m+m3^n)}.$$

Problem 4.25. $(*)$ *Compute*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{mn(m+n+1)}.$$

What about

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{mnk(m+n+k+1)}?$$

Can you generalise this to more variables?

Problem 4.26. $(*)$ *Compute*

$$\sum_{n=0}^{\infty} \cos^{-1}(n^2+n+1),$$

where $\cos^{-1}(t)$ *for* $t>0$ *is defined as the unique* $\theta \in (0,\pi/2]$ *such that* $\cos(\theta)=t$.

Problem 4.27. $(*)$ *Determine the value of the double sum*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n+1}.$$

Problem 4.28. $(*)$ *Let* $\alpha(n)$ *denote the number of zeroes in the expansion of* n *in base 3. For which positive* x *does the series*

$$\sum_{n=1}^{\infty} \frac{x^{\alpha(n)}}{n^3}$$

converge?

4.5 IMC Problems

Problem 4.29. (1995)($*$) *Let* (b_n) *be the sequence of real numbers defined by* $b_0 = 1$ *and* $b_{n+1} = 2 + \sqrt{b_n} - 2\sqrt{1+\sqrt{b_n}}$. *Compute*

$$\sum_{n=1}^{\infty} b_n 2^n.$$

Problem 4.30. (1996)($*$) *Show that*

$$\lim_{x\to\infty} \sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} = \frac{1}{2}.$$

Problem 4.31. (1997)($*$) *Let* (a_n) *be a sequence of positive numbers with* $\lim_{n\to\infty} a_n = 0$. *Determine*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log\left(\frac{k}{n} + a_n\right).$$

Problem 4.32. (1997) *Show that*

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sin(\log(n))}{n^{\alpha}}$$

converges if and only if $\alpha > 0$.

Problem 4.33. (1999) *Let* $\pi : \mathbb{N} \to \mathbb{N}$ *be a bijection. Show that the following sum diverges:*

$$\sum_{n=1}^{\infty} \frac{\pi(n)}{n^2}.$$

Problem 4.34. (2001) *Determine*

$$\lim_{t\to 1} (1-t) \sum_{n=1}^{\infty} \frac{t^n}{1+t^n}.$$

Problem 4.35. (2003)($*$) *Let* (a_n) *be the sequence defined by* $a_0 = 1$ *and*

$$a_{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} \frac{a_k}{n-k+2}.$$

Find the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{a_k}{2^k}.$$

Problem 4.36. (2010)($*$) *Determine*

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)}.$$

Problem 4.37. (2011)($*$) *Determine the value of*

$$\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right)\log\left(1+\frac{1}{2n}\right)\log\left(1+\frac{1}{2n+1}\right)$$

Problem 4.38. (2012)($*$) *Define a sequence* (a_n) *by setting* $a_0 = 1$, $a_1 = 1/2$ *and further*

$$a_{n+1} = \frac{na_n^2}{1+(n+1)a_n} \quad \textit{for all } n \geqslant 1.$$

Show that the series $\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$ *converges and determine its limit.*

Problem 4.39. (2014)($*$) *Consider the sequence*

$$(a_n) = (1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots)$$

Determine all pairs (α, β) *of positive real numbers such that*

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} \sum_{k=1}^{n} a_k = \beta.$$

Problem 4.40. (2015) *Define a sequence* a_n *by* $a_0 = 0$, $a_1 = 3/2$ *and* $a_{n+2} = \frac{5}{2}a_{n+1} - a_n$ *for all* $n \geqslant 0$. *Determine*

$$\sum_{n=0}^{\infty} \frac{1}{a_{2^n}}.$$

Problem 4.41. (2015) *Prove that*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(n+1)} < 2.$$

Problem 4.42. (2016) *Let* (x_n) *be a sequence of positive real numbers such that*

$$\sum_{n=1}^{\infty} \frac{x_n}{2n-1} = 1.$$

Show that

$$\sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{x_n}{k^2} \leqslant 2.$$

Problem 4.43. (2018) *Let* (a_n) *and* (b_n) *be two sequences of positive numbers. Show that*

$$\sum_{n=1}^{\infty}\sqrt{\frac{a_n}{b_n}}<\infty \iff \text{ there exists } (c_n) \text{ such that } \sum_{n=1}^{\infty}\frac{a_n}{c_n}<\infty \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{c_n}{b_n}<\infty.$$

Problem 4.44. (2018) *Let* (a_n) *be defined by* $a_0=0$ *and* $a_{n+1}^3=a_n^2-8$*. Prove that*

$$\sum_{n=0}^{\infty}|a_{n+1}-a_n|<\infty.$$

Problem 4.45. (2019) *Evaluate the product*

$$\prod_{n=3}^{\infty}\frac{(n^3+3n)^2}{n^6-64}.$$

5 Algebra II

We continue with a brief treatment of some elementary number theory. IMC problems rarely require advanced knowledge of this very deep and active field, so we will content ourselves with only some very basic concepts.

5.1 Number Theory

Number theory, in its simplest terms, describes the study of the natural numbers and their properties. We list some very basic ones.

- If $d \mid m$ and $d \mid n$, then also $d \mid am + bn$ for any $a, b \in \mathbb{Z}$.
- If $d \mid m$ and $m \mid n$, then also $d \mid n$.
- If p is a prime and $p \mid mn$, then $p \mid m$ or $p \mid n$.
- Every positive integer has a unique prime factorisation.
- If $\gcd(m, n) = 1$, then $\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. *(Chinese Remainder Theorem)*

A very important arithmetic function is the *Euler-totient function* $\varphi : \mathbb{N} \to \mathbb{N}$, which is defined by $\varphi(n) = \#(\mathbb{Z}/n\mathbb{Z})^\times$. The following properties are easy to show, yet very important.

- If p is a prime and $k \in \mathbb{N}$, then $\varphi(p^k) = p^{k-1}(p-1)$.
- If $\gcd(m, n) = 1$, then by the CRT, $\varphi(mn) = \varphi(m)\varphi(n)$.
- For any $n \in \mathbb{N}$ we have the direct formula

$$\varphi(n) = n \prod_{p \mid n} \frac{p-1}{p}.$$

The following result is a direct result from Lagrange's Theorem in group theory.

Proposition 5.1. (Fermat's Little Theorem) *If* $\gcd(a, n) = 1$, *then* $a^{\varphi(n)} \equiv 1 \mod n$. *In particular, for primes* p, *it holds that* $a^p \equiv a \mod p$ *for all* $a \in \mathbb{Z}$.

Finally, the following is almost immediate from the fact that $(\mathbb{Z}/p\mathbb{Z})^\times$ is a group.

Proposition 5.2. (Wilson) *For any prime number* p, *it holds that* $(p-1)! \equiv -1 \mod p$.

To solve number theory problems, it is very often a good idea to apply modular arithmetic; especially powers of numbers often display great regularity when considering the right modulus. It is also useful to keep the Legendre symbol in mind;

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \mod p \in \{-1, 0, 1\}.$$

It decides whether or not a number is a quadratic residue modulo p or not. Its most striking property is quadratic reciprocity.

Theorem 5.3. *Let p and q be two distinct odd primes. Then*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4}.$$

The prime 2 is a tad more subtle; $\left(\frac{2}{p}\right)$ is equal to 1 precisely when $p \equiv \pm 1 \mod 8$.

5.2 Examples

Example 5.4. Let p be a prime and let $d > 1$ be a divisor of $2^p - 1$. Prove that $d > p$.

Solution: It suffices to show that every prime divisor q of $2^p - 1$ is larger than p. To this end, suppose that $q \mid 2^p - 1$, or in other words, that $2^p \equiv 1 \mod q$. This means that the order of the element $2 \mod q$ in the group $(\mathbb{Z}/q\mathbb{Z})^\times$ divides p. Since it is not 1, this order must equal p. But this order must also divide the order of the group itself. We find that $p \mid q - 1$, and so in particular $p < q$; this completes the proof. △

Example 5.5. Determine all pairs of integers $x, y \in \mathbb{N}$ such that

$$1! + 2! + \ldots + x! = y^2.$$

Solution: Note that $5 \mid x!$ as soon as $x \geqslant 5$. This implies that for $x \geqslant 4$,

$$1! + 2! + \ldots + x! \equiv 1 + 2 + 6 + 24 + 0 + \ldots + 0 = 33 \equiv 3 \mod 5.$$

However, 3 is not a square modulo 5, so we find no solutions for $x \geqslant 4$. We manually check the cases $x \leqslant 3$, to find only the pairs $(1, 1)$ and $(3, 3)$ as solutions. △

Example 5.6. Let $m, n \in \mathbb{N}$. Prove that $4mn - m - n$ is not a square.

Solution: Suppose that $4mn - m - n = k^2$ for some integer k. Then we find that

$$16mn - 4m - 4n = 4k^2 \implies (4m - 1)(4n - 1) = (2k)^2 + 1.$$

We claim that any prime number $p \mid (2k)^2 + 1$ must be $1 \mod 4$. This would complete the proof, as this would show that it is impossible for $(2k)^2 + 1$ to admit a factor $4m - 1 \equiv -1 \mod 4$. To show the claim, we note that for such a prime p, it holds that $(2k)^2 \equiv -1 \mod p$. This means that -1 is a square modulo p, and as such, the claim follows from the theory about Legendre symbols above. △

Example 5.7. How many primes are of the form $3711\ldots11$?

Solution: Let $a_n = 3711\ldots11$, containing precisely n ones at the end.

Our first observation will be that $111 = 3 \cdot 37$. Appending three ones to a number k is equivalent to the operation $k \mapsto 1000k + 111$. In particular, divisibility by 3 and 37 is preserved. As $37 \mid a_0$, this shows that $37 \mid a_n$ as soon as $3 \mid n$. Similarly, by considering the sum of its digits, one finds that $3 \mid 3711 = a_1$, and as such, $3 \mid a_n$ as soon as $n \equiv 2 \mod 3$. We thus reduce to analysing the case that $n \equiv 1 \mod 3$. We look for small prime factors for these remaining numbers, to find

$$371 = 7 \cdot 53, \quad \text{and} \quad 371111 = 13 \cdot 28547.$$

Could similar patterns as above exist for these primes? Note that

$$111111 = 1001 \cdot 111 = (7 \cdot 11 \cdot 13) \cdot (3 \cdot 37).$$

With similar reasoning as above, we find that $7 \mid a_n \implies 7 \mid a_{n+6}$ and similarly $13 \mid a_n \implies 13 \mid a_{n+6}$. Our calculations show that $7 \mid a_1$ and $13 \mid a_4$, and as such, we have identified a prime factor of every a_n. So, such numbers can never be prime. $\triangle$

5.3 Exercises

Problem 5.1. *Determine all* $n \in \mathbb{Z}$ *such that* $n^2 + 1$ *is divisible by* $n + 1$.

Problem 5.2. *Let* $p > 3$ *be a prime. Show that* $p^2 - 1$ *is divisible by 24.*

Problem 5.3. *Is it possible for a power of 2 to end in the digits 2012?*

Problem 5.4. *Determine all prime numbers* p *for which there exist primes* q, q' *and* r, r' *such that* $p = q + q' = r - r'$.

Problem 5.5. *Find the largest positive integer* m *with the property that for any* $n \in \mathbb{N}$, *the number* $n(n+1)(2n+1)$ *is divisible by* m.

Problem 5.6. *Let* $n \geqslant 2$ *be an integer. What is the final digit of the Fermat-number* $2^{2^n} + 1$?

Problem 5.7. *Determine all pairs of integers* $k, \ell \in \mathbb{N}$ *such that* $k^2 = 2^\ell + 3$.

Problem 5.8. *Prove that for any positive integer* n, *the number* $n^2 + 2n + 12$ *is not divisible by* 121.

Problem 5.9. *An* $m \times n$ *grid of unit squares has the property that the border consist of precisely 8% of the total number of squares. What possible areas can the board have?*

Problem 5.10. *Let* $p \geqslant 5$ *be a prime number. Prove that the number consisting of* $p - 1$ *ones in a row is divisible by* p.

Problem 5.11. *Let* $a, b \in \mathbb{N}$ *be coprime. Prove that* $\gcd(a + b, a^2 - ab + b^2) \in \{1, 3\}$.

Problem 5.12. *Determine all quintuples of primes* p, q, r, s, t *such that* $p^2 + q^2 = r^2 + s^2 + t^2$.

Problem 5.13. *Determine all pairs of positive integers* $x, y \in \mathbb{Z}$ *such that* $1/x + 1/y = 1/14$.

Problem 5.14. *Determine all integer solutions to the equation* $x^{25} - y^{12} = x + 3$.

Problem 5.15. *Prove that for any* $n \geqslant 2$, *the number* $2^n - 1$ *is not divisible by* n.

Problem 5.16. *Prove that every positive composite integer can be written as* $xy + yz + zx + 1$ *for some positive integers* x, y, z.

Problem 5.17. *Prove that*

$$\frac{\gcd(m, n)}{n}\binom{n}{m}$$

is an integer for any $n \geqslant m \geqslant 1$.

Problem 5.18. *Let* $n = 9^{2023}$. *Determine* $\gcd(n^2 + 2, n^3 + 1)$.

Problem 5.19. *Determine all pairs of integers* $x, y \in \mathbb{Z}$ *such that* $x^4 + y^3 = 2613527$.

Problem 5.20. *Let* $u, v \in \mathbb{N}$ *be coprime integers. Prove that there exist integers* $a, b > 1$ *and* $m, n \in \mathbb{N}$ *such that* $a^{n^u} = b^{m^v}$ *and* $a^u = b^v$.

Problem 5.21. *Determine all Fibonacci numbers that are also of the form* $2^{2^m} + 1$.

Problem 5.22. *Determine the largest proper divisor of* $2015^5 + 2035^4 + 1$.

Problem 5.23. *Determine all pairs of positive integers* $a, b \in \mathbb{N}$ *such that* $a^2 + 2b^2 = 2023$.

Problem 5.24. *Determine all pairs of positive integers such that* $3n^2 + 3n + 7 = m^3$.

Problem 5.25. *Suppose that* $\{a_1, \ldots, a_n\}$ *and* $\{b_1, \ldots, b_n\}$ *are a set of representatives for the classes of* $\mathbb{Z}/n\mathbb{Z}$. *If also* $\{a_1b_1, \ldots, a_nb_n\}$ *is such a set, prove that* $n \in \{1, 2\}$.

Problem 5.26. *Find all right-angled triangles with integral side lengths* $a < b < c$ *such that* $c - b = b - a$.

Problem 5.27. *How many numbers of the form* $10101\ldots101$ *are prime?*

Problem 5.28. *Show that for any positive integer* n, *it holds that*

$$n! = \prod_{i=1}^{n} \operatorname{lcm}(1, 2, \ldots, \lfloor n/i \rfloor).$$

Problem 5.29. *Determine all triples of integers* $x, y, z \in \mathbb{Z}$ *satisfying* $x^2 + 10y^2 = 3z^2$.

Problem 5.30. *Determine all quadruples of positive integers* (m, n, x, y) *with* $\gcd(n, m) = 1$ *further satisfying* $(x^2 + y^2)^m = (xy)^n$.

Problem 5.31. *Let* p *be an odd prime. Prove that the function*

$$F : \mathbb{Z}/p\mathbb{Z} : n \mapsto 1 + 2n + 3n^2 + \ldots + (p-1)n^{p-2}$$

is bijective.

Problem 5.32. *Let* $S = \mathbb{Q} \setminus \{0, \pm 1\}$*. Let* $f : S \to S$ *be given by* $f(x) = x - 1/x$*. Prove that*

$$\bigcap_{n=1}^{\infty} f^n(S) = \varnothing.$$

Problem 5.33. *Determine all pairs of* $a, n \in \mathbb{N}$ *such that* $a^{n+1} - (a+1)^n = 2001$*.*

Problem 5.34. *Define the sequence* (a_n) *through* $a_0 = 1$ *and further* $a_{2n+1} = a_n$ *and* $a_{2n+2} = a_n + a_{n+1}$ *for all* $n \geqslant 0$*. Prove that every positive rational number can be written as* a_n/a_{n+1} *for some* $n \geqslant 0$*.*

Problem 5.35. *Let three positive integers be written on a blackboard. A move consists of choosing two of the numbers on the board, say* x *and* y*, wiping them out and replacing them by* $2x$ *and* $y - x$*. Prove that it is possible to at some point write a* 0 *on the board.*

Problem 5.36. *Let* S *be a finite set of integers exceeding 1. Suppose that for any integer* n*, there exists some* $s \in S$ *such that either* $\gcd(s, n) = 1$ *or* $\gcd(s, n) = s$*. Prove that there exist* $s, t \in S$ *such that* $\gcd(s, t)$ *is prime.*

Problem 5.37. *Determine all positive integers* n *with the property that if* a *and* b *are positive integers such that* $1/n = 1/a + 1/b$*, then either* $a \mid b$ *or* $b \mid a$*.*

Problem 5.38. *Determine the remainder upon dividing* 2^{2^n} *by* $2^n - 1$ *if* n *itself is also a power of 2. What if* n *is a prime?*

Problem 5.39. *Let* p *be a prime. Determine* $\binom{2p}{p} \mod p$*.*

Problem 5.40. *Determine all pairs of integers* $k, m \in \mathbb{Z}$ *such that* $3 \cdot 2^k = m^3 + 5m + 6$*.*

Problem 5.41. *Determine for every prime* p *an explicit integer* n *such that* $2^n + 3^n + 6^n - 1$ *is divisible by* p*.*

Problem 5.42. *Let* m *and* a *be positive integers such that* $a^5 + 1$ *is divisible by* m*. Prove that either* $a + 1$ *is divisible by* m*, or that* $\varphi(m)$ *is divisible by 5.*

Problem 5.43. *Does there exist a positive integer* n *with the property that* $103 \mid n$ *and* $2^{2n+1} \equiv 2 \mod n$*?*

Problem 5.44. *Let* p *and* $q \neq 5$ *be primes such that* $q \mid 2^p + 3^p$*. Prove that* $q > 2p$*.*

Problem 5.45. *Is* $712! + 1$ *a prime?*

Problem 5.46. *Let* n *be a positive integer, let* p *be a prime and let* d *be a divisor of the number* $(n+1)^p - n^p$*. Prove that* $d - 1$ *is divisible by* p*.*

Problem 5.47. *Let $m, n \geqslant 3$ be odd integers. Prove that $2^m - 1$ is not a divisor of $3^n - 1$.*

Problem 5.48. *Let $x, y \geqslant 2$ be positive integers with $\gcd(x, y) = 1$. Prove that $x^7 + y^7$ is divisible by 7 or by a prime that is 1 modulo 7.*

Problem 5.49. *Starting with the number 7^{1996}, we repeatedly remove the first digit and add it to the remaining number. Prove that, when we have reached a 10 digit number, at least two digits are equal.*

Problem 5.50. *Prove that for any $k \geqslant 1$, the number $512 \cdot 12^k + 1$ is composite.*

Problem 5.51. *The number 2^{29} is a 9-digit number of which it is known that all digits are distinct. Which digit does not occur?*

Problem 5.52. *Determine all positive integers that cannot be written as the sum of two or more consecutive positive integers.*

Problem 5.53. *How many prime numbers are of the form $3811\ldots11$?*

Problem 5.54. *Determine all pairs of integers x, y satisfying $2x^6 + y^7 = 11$.*

Problem 5.55. *Prove that $n^n + (n+1)^{n+1}$ is composite for infinitely many choices of n.*

Problem 5.56. *Prove that the number 10101 is composite in every possible base $b \geqslant 2$.*

Problem 5.57. *Determine all positive even numbers that can be written as the sum of two composite odd numbers.*

Problem 5.58. *Is the set of positive integers n such that $2^n - 8$ is divisible by n finite or infinite?*

Problem 5.59. *Define the sequence (a_n) by $a_0 = 1$ and $a_{n+1} = 2^{a_n}$ for every $n \geqslant 0$. Prove that $a_n \equiv a_{n-1} \mod n$ for every $n \geqslant 2$.*

Problem 5.60. *Prove that there exist infinitely many composite n such that $3^{n-1} - 2^{n-1}$ is divisible by n.*

Problem 5.61. *Let A be the sum of the decimal digits of 4444^{4444} and let B be the sum of the digits of A. Determine the sum of the digits of B.*

Problem 5.62. *Let $a, b, c \in \mathbb{N}$ be such that $a^2 - bc$ is a square. Prove that $2a + b + c$ is not a prime.*

Problem 5.63. *Prove that for every positive integer n, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not a prime.*

Problem 5.64. *Consider the set $S = \{2^k - 3 \mid k \in \{2, 3, \ldots\}\}$. Prove that there exists an infinite subset $T \subset S$ such that any two elements of T are coprime.*

Problem 5.65. $(*)$ *Let $n > 1$ be an integer such that $n \mid 3^n + 4^n$. Prove that $7 \mid n$.*

Problem 5.66. $(*)$ *Determine all primes p such that the number of solutions $x, y \in \mathbb{Z}$ with $0 \leqslant x, y < p$ to the equation $y^2 = x^3 - x \mod p$ is precisely equal to p.*

5.4 IMC Problems

Problem 5.67. (1994) *Let $n \in \mathbb{N}$ and let S be a set of $2n-1$ distinct irrational numbers. Prove that there are n distinct elements $x_1, \ldots, x_n \in S$ such that for all non-negative rational numbers $a_1, \ldots, a_n$ with $a_1 + \ldots a_n > 0$, we have that $a_1 x_1 + \ldots + a_n x_n$ is an irrational number.*

Problem 5.68. (1996) *Let y be a real number for which $\cosh(y)$ is an integer. Show that $\cosh(ny)$ is an integer for any $n \in \mathbb{N}$. Use this to show that if $\cosh(ny)$ and $\cosh((n+1)y)$ are both rational for some $n \geqslant 1$, then $\cosh(my)$ must be rational for all $m \in \mathbb{N}$.*

Problem 5.69. (1997)($*$) *Let $\alpha \in (1, 2)$ be a real number. Show that α has a unique representation as an infinite product*

$$\alpha = \left(1 + \frac{1}{n_1}\right)\left(1 + \frac{1}{n_2}\right)\cdots$$

where each n_i is a positive integer satisfying $n_i^2 \leqslant n_{i+1}$ for each $i \geqslant 1$. Show further that α is rational if and only if for some $m \geqslant 1$ and all $k \geqslant m$, it holds that $n_{k+1} = n_k^2$.

Problem 5.70. (2003) *Determine the set of all pairs (a, b) of positive integers for which the set $\mathbb{N}$ of positive integers can be decomposed into two sets A and B such that $a \cdot A = b \cdot B$.*

Problem 5.71. (2006) *Find the number of positive integers $x < 10^{2006}$ such that $x^2 - x$ is divisible by 10^{2006}.*

Problem 5.72. (2006) *Prove that there exists an infinite number of relatively prime pairs (m, n) of positive integers such that the equation $(x+m)^3 = nx$ has three distinct integer roots.*

Problem 5.73. (2007) *Let $x, y, z \in \mathbb{Z}$ be such that $S = x^4 + y^4 + z^2$ is divisible by 29. Prove that S is even divisible by 29^4.*

Problem 5.74. (2008) *Let n be a positive integer. Prove that 2^{n-1} divides*

$$\sum_{0 \leqslant k < n/2} \binom{n}{2k+1} 5^k.$$

Problem 5.75. (2010) *Let $a, b \in \mathbb{Z}$ and suppose that n is a positive integer for which the set $\mathbb{Z} \setminus \{ax^n + by^n \mid x, y \in \mathbb{Z}\}$ is finite. Prove that $n = 1$.*

Problem 5.76. (2012) *Determine whether or not the set of positive integers n for which $n! + 1$ is a divisor of $(2012n)!$ is finite or infinite.*

Problem 5.77. (2013) *Let $p, q \in \mathbb{N}$ satisfy $\gcd(p, q) = 1$. Prove that*

$$\sum_{k=0}^{pq-1} (-1)^{\lfloor k/p \rfloor + \lfloor k/q \rfloor} = \begin{cases} 0 & \text{if } pq \text{ is even;} \\ 1 & \text{if } pq \text{ is odd.} \end{cases}$$

Problem 5.78. (2013)($*$) *Does there exist an infinite subset $M \subset \mathbb{N}$ such that for any $a, b \in M$, the sum $a + b$ is square-free?*

Problem 5.79. (2014) *Let $n > 6$ be a perfect number and let $n = p_1^{e_1} \cdots p_k^{e_k}$ be its prime factorisation with $p_1 < \ldots p_k$. Prove that e_1 must be even.*

Problem 5.80. (2014) *For a positive integer x, let $d_n(x)$ denote its nth decimal digit. Suppose that for some sequence (a_n), there are only finitely many zeroes in the sequence $(d_n(a_n))$. Prove that there are infinitely many positive integers that do not occur in the sequence (a_n).*

Problem 5.81. (2015) *For a positive integer n, let $f(n)$ be the number obtained by writing n in binary and replacing every 0 with 1 and vice versa. Prove that $4\sum_{k=1}^{n} f(k) \leqslant n^2$. When does equality hold?*

Problem 5.82. (2015) *Determine whether or not there exist 15 integers $m_1, \ldots, m_{15}$ such that*

$$\sum_{k=1}^{15} m_k \cdot \arctan(k) = \arctan(16).$$

Problem 5.83. (2017) *For any positive integer m, let $P(m)$ denote the product of the positive divisors of m. For every positive integer n, define the sequence $a_1(n), a_2(n), \ldots$ through $a_1(n) = n$ and $a_{k+1}(n) = P(a_k(n))$. Prove that for every subset $S \subset \{1, 2, \ldots, 2007\}$, there exists a positive integer n such that for every $1 \leqslant k \leqslant 2007$, the number $a_k(n)$ is a perfect square if and only if $k \in S$.*

Problem 5.84. (2018)($*$) *Let $p < q$ be prime numbers. Suppose that in a convex polygon $P_1P_2 \ldots P_{pq}$ all angles are equal and all side lengths are distinct positive integers. Prove that $|P_1P_2| + |P_2P_3| + \ldots + |P_kP_{k+1}| \geqslant \frac{k^3+k}{2}$ for all integers $1 \leqslant k \leqslant p$.*

Problem 5.85. (2019) *Let C denote the set of composite integers. For each $n \in C$, let a_n denote the smallest positive integer k such that $k!$ is divisible by n. Determine whether the following series converges:*

$$\sum_{n \in C} \left(\frac{a_n}{n}\right)^n$$

Problem 5.86. (2020) *Find all prime numbers p for which there exists a unique integer $1 \leqslant a \leqslant p$ such that $a^3 - 3a + 1$ is divisible by p.*

Problem 5.87. (2022) *Let $p > 2$ be a prime number. Prove that there is a permutation $(x_1, x_2, \ldots, x_{p-1})$ of $\{1, 2, \ldots, p-1\}$ such that $x_1x_2 + x_2x_3 + \ldots + x_{p-2}x_{p-1} \equiv 2 \mod p$.*

Problem 5.88. (2023)($*$) *Let p be a prime number and let k be a positive integer. Suppose that the numbers $a_i = i^k + i$ for $i = 0, 1, \ldots, p-1$ for a complete residue system modulo p. What is the set of possible remainders of a_2 upon division by p?*

Problem 5.89. (2023)($*$) *For every positive integer n, let $f(n)$ and $g(n)$ be the minimal positive integers such that*

$$1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} = \frac{f(n)}{g(n)}.$$

Determine whether there exists a positive integer n for which $g(n) > n^{0.999n}$.

6 Combinatorics I

Combinatorics is one of the most represented topics on competitions like the IMO, but also the IMC is no stranger to interesting counting problems or strange mathematical games. This chapter will feature little theory; most of the techniques are completely elementary, as is typical for combinatorics, but this challenging field is not to be underestimated in terms of its depth, richness and complexity.

6.1 Basic counting

Many problems in combinatorics are as simple as counting the number of ways something can be done, or the number of objects satisfying a certain set of properties. Many different techniques can be used to approach these kinds of problems, of which we list a few below that may offer some inspiration in case the reader needs some while working on the exercises below.

- Always try to make the problem more manageable first by plugging in small values of the parameters that occur in the problem. This often allows you to play around with things and can often be very helpful in developing a conjecture or a more profound understanding of the problem.
- If a setting of a combinatorial problem is dependent on the choice of some $n \in \mathbb{N}$, it is often inviting to try induction. This idea is particularly strong if for $k < n$, the conclusion to the problem for k gives some information about the situation for n.
- Sometimes counting the same set in two different ways can give meaningful information. Recall the *handshaking lemma*: in a group of n people, some of which shook each other's hand, the number of people who have shaken an odd number of hands must be even. Indeed, this is equivalent to saying that the total number of handshakes is even. Instead of adding up the contributions from each individual, we recognise that adding up the contributions from each handshake should give the same result; the claim follows.
- Looking for bijections can often be useful. Proving that a set S has even cardinality is implied by the existence of a involution $S \to S$ without fixed points, or equivalently, by disjoint subsets $A, B \subset S$ with a bijection $A \to B$. Alternatively, a bijection from S to a set S' whose objects are defined differently can sometimes make things easier to count.

6.2 Advanced counting

It is important to be handy with binomial coefficients; they pop up in many different settings and problems, but the most important properties to remember are

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad \text{and} \quad (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

The following is informally (and perhaps only locally) known as the *Easter-egg principle*.

Proposition 6.1. *Consider a set of n easter eggs. We wish to paint each egg in one of k different colours. Then the number of different possible colour combinations is given by*

$$\binom{n+k-1}{k-1}.$$

Sometimes the most natural way to count a certain quantity is to build it up from smaller cases of itself. Typically, this results in a recursion relation. Not every recursion relation is solvable in a closed form, but sometimes they are. Writing out small values can sometimes give you a strong idea for as to how the sequence behaves and might lead to a guess for a closed form. In the special case of a *linear recursion*, it is always possible to find a closed form.

Proposition 6.2. *If a sequence (a_n) is defined by the initial values $a_1, \ldots, a_m$ and further*

$$a_n = c_1 a_{n-1} + \ldots + c_m a_{n-m} \quad \textit{for all } n > m,$$

then define the polynomial

$$f(X) = X^m - c_1 X^{m-1} - \ldots - c_m.$$

Let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ be its roots and suppose that they are all distinct. Then there exist unique $C_1, \ldots, C_m \in \mathbb{C}$ such that

$$a_n = C_1 \lambda_1^n + \ldots + C_m \lambda_m^n \quad \textit{for all } n \geqslant 1.$$

Typically, one applies this result to solve two step recursions, because in general, the numbers $\lambda_1, \ldots, \lambda_n$ might be rather nasty. The largest absolute value of these roots determines the growth of the sequence and even for real sequences, it is possible for some of the λ_i to be complex. This result is still useful even when the λ_n turn out to be integers; the sequence

$$a_n = 5a_{n-1} - 6a_{n-2} \quad \text{with } a_0 = 0 \text{ and } a_1 = 1$$

starts off as $0, 1, 5, 19, 65, \ldots$, from which the general formula $a_n = 3^n - 2^n$ is not easily guessed. The above result yields it very quickly, though.

6.3 Examples

Example 6.3. *(IMC 2012)* For any positive integer n, we let $p(n)$ denote the number of different ways to write n as the sum of positive integers. For example, because

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1,$$

it holds that $p(4) = 5$. Prove that for any $n \geqslant 2$, the number of ways to write n as the sum of integers which are all strictly greater than 1 is precisely equal to $p(n) - p(n-1)$.

Solution: Let P_n denote the set of ways we can write n as the sum of positive integers, and S_n the set of ways of doing so without using the number 1. We construct a bijection $P_n \to P_{n-1} \cup S_n$ from which the desired conclusion would immediately follow. Indeed, any partition in P_n either contains a 1, or it does not. In the former case, we remove this 1 to end up with an element from P_{n-1}. In the latter case, this partition describes an element from S_n by definition. The inverse map is obvious; to a partition in P_{n-1} we add a 1 back, and to an element from S_n we do nothing. The bijection has been established. △

Example 6.4. *(IMC 2022)* We colour all the sides and diagonals of a regular polygon P with 43 vertices either red or blue in such a way that every vertex is an endpoint of 20 red segments and 22 blue segments. A triangle formed by vertices of P is called monochromatic if all of its sides have the same colour. Suppose that there are 2022 blue monochromatic triangles. How many red monochromatic triangles are there?

Solution: Define a *corner* to be a set of two distinct edges from K_{43} that have a vertex in common. We observe that a monochromatic triangle always contains three monochromatic corners, and that a polychromatic triangle always contains one monochromatic corner and two polychromatic corners. Therefore we study the quantity $2M - P$, where M is the number of monochromatic corners and P is the number of polychromatic corners. By observing that every corner is part of a unique triangle, we can split this quantity up into all the distinct triangles in K_{43}. By construction the contribution of a polychromatic triangle will vanish, whereas a monochromatic triangle will contribute 6. We conclude that

$$2M - P = 6\#\{\text{monochromatic triangles}\}.$$

Consider any vertex v. Let M_v be the number of monochromatic corners with central vertex v and P_v the number such polychromatic corners. It then follows that

$$M_v = \frac{20 \cdot 19}{2} + \frac{22 \cdot 21}{2} = 421 \quad \text{and} \quad P_v = 20 \cdot 22 = 440.$$

In other words, for any vertex v it holds that $2M_v - P_v = 402$. Adding up all these contributions, we find that

$$2M - P = 43 \cdot 402.$$

We conclude that there are $43 \cdot 402/6 = 43 \cdot 67 = 2881$ monochromatic triangles in total. Since 2022 of these were blue, 859 must be red. △

6.4 Exercises

Problem 6.1. *How many subsets of* $\{1, 2, \ldots, n\}$ *consist of an even number of elements?*

Problem 6.2. *For* $n \geqslant 2$*, let* T_n *denote the number of non-empty subsets* S *of* $\{1, 2, \ldots, n\}$ *such that the mean of the numbers occurring in* S *is an integer. Prove that* $T_n + n$ *is even.*

Problem 6.3. *In how many ways can be write* 2023 *as the sum of 42 non-negative integers? What about positive integers? What about positive odd integers?*

Problem 6.4. *A pawn is placed on an infinite square grid. A step consists of the pawn moving to one of the four squares that touch its current square. Given* $n \geqslant 1$*, how many routes can the pawn take that consist of* $2n$ *steps but bring him back to where he started?*

Problem 6.5. *How many permutations of* $\{1, 2, \ldots, n\}$ *can we find such that every number is either smaller or bigger than all numbers that came before it?*

Problem 6.6. *Let* $k, n, \ell \in \mathbb{N}$*. How many subsets of* $\{1, 2, \ldots, n\}$ *are there of size* k *such that for any* $a, b \in S$ *it holds that* $|a - b| > \ell$*?*

Problem 6.7. *How many sequences of length* n*, consisting of only zeroes and ones, can be make such that we never write down three zeroes or three ones in a row?*

Problem 6.8. *Let* $n, k \in \mathbb{N}$*. Determine*

$$\sum_{j=0}^{k} \binom{k}{j}^2 \binom{n+2k-j}{2k}.$$

Problem 6.9. *We call a subset* $S \subset \{1, 2, \ldots, n\}$ mediocre *if* S *satisfies the property that if* $a, b \in S$ *and* $a + b$ *is even, then also* $(a + b)/2 \in S$*. Let* $A(n)$ *denote the number of mediocre subsets of* $\{1, 2, \ldots, n\}$*. Find all* $n \geqslant 1$ *such that* $A(n+2) + A(n) = 2A(n+1) + 1$*.*

Problem 6.10. *A group of* $2n$ *students, all with different heights, must pose for a photo. They must arrange themselves into two lines of* n *students such that in each line, the heights increase from left to right, and such that every person on the back row is taller than the person directly in front of them. In how many ways can they arrange themselves?*

Problem 6.11. *Define the sequence* (c_n) *by the rules* $c_1 = 1$ *and further* $c_{2n} = c_n$ *and* $c_{2n+1} = (-1)^n c(n)$*. Determine*

$$\sum_{n=1}^{2023} c(n)c(n+2).$$

Problem 6.12. *For a positive integer* n*, let* $C(n)$ *denote the number of ways to write* n *as the sum of non-increasing powers of 2, such that no power of 2 is used more than three times. Determine* $C(n)$*.*

Problem 6.13. *Let a rook travel on an* $n \times 3$ *chessboard. How many routes can the rook take from the bottom-left square to the bottom-right square such that the rook passes each square exactly once?*

Problem 6.14. *We call a finite set of numbers* selfish *if it contains its own cardinality. Determine the number of selfish subsets of* $\{1, 2, \ldots, n\}$ *that contain no other selfish sets.*

Problem 6.15. *A Dyck-path of length* n *is a path with steps* $(1, 1)$ *and* $(1, -1)$ *that starts at* $(0, 0)$ *and ends at* $(2n, 0)$ *that never crosses the* x*-axis. A* return *of such a path is a maximal sequence of steps in the same direction that ends on the* x*-axis. Prove that that there is a bijection between Dyck-paths of length* n *without returns of even length, and Dyck-paths of length* $n - 1$*.*

Problem 6.16. *For a word* W *using only the letters* A *and* B*, we let* $\Delta(W)$ *denote the number of* A*'s minus the number of* B*'s. We say such a word is* balanced *if for every subword* W' *of* W*, it holds that* $\Delta(W') \leqslant 2$*. Determine the number of balanced words of length* n*.*

Problem 6.17. *For positive integers* n *and* m*, we define* $f(n, m)$ *as the number of* n*-tuples* $(x_1, \ldots, x_n)$ *of integers such that* $|x_1| + \ldots + |x_n| \leqslant m$*. Prove that* $f(n, m) = f(m, n)$*.*

Problem 6.18. *Let* $k < n$ *be positive integers. Show that the number of ways to write* n *at the sum of precisely* k *positive integers is equal to the number of ways to write* n *as the sum of positive integers such that* k *is the largest of the numbers in the partition.*

Problem 6.19. *Given* $k < n$ *positive integers, determine the number of subsets of* $\{1, 2, \ldots, n\}$ *of size* k *containing no two consecutive numbers.*

Problem 6.20. $(*)$ *Let* $n \geqslant 3$ *and consider a circle with* $n + 1$ *marked points on it, among which one special point. We bijectively assign a label from the set* $\{0, 1, \ldots, n\}$ *to each marked point, ensuring that the special point is labelled* 0*. We call such a labelling* pretty *if for any* $0 \leqslant a < b < c < d \leqslant n$ *with* $a + d = b + c$*, the chord between the points labelled with* a *and* d *does not intersect the chord between the points labelled* b *and* c*. Let* M *be the number of pretty labellings, and let* N *denote the number of ordered pairs* (x, y) *such that* $x, y > 0$*, gcd*$(x, y) = 1$ *and* $x + y \leqslant n$*. Prove that* $M = N + 1$*.*

Problem 6.21. $(*)$ *Let* P *be the set of paths from* $(0, 0)$ *to* (n, n) *that consist of steps of length 1 either up or to the right. For any path* $p \in P$*, let* $a(p)$ *denote the number of points on* p *of the form* (i, i) *for* $0 \leqslant i \leqslant n$*. Determine*

$$\sum_{p \in P} a(p).$$

Problem 6.22. $(*)$ *Let* S *be a finite set containing* m *elements. For a function* $f : S \to S$*, let* f^n *denote the* n*-fold composition of* f *with itself. We say that* f *is* boring *if* $f^n = f^{n+1}$ *for some* $n \geqslant 1$*. Determine the number of boring functions.*

Problem 6.23. $(*)$ *For any positive integer* n*, let* $p(n)$ *denote the set of distinct ways to write* n *as the sum of positive integers. For each partition* $\pi \in p(n)$*, let* $S(\pi)$ *denote the number of positive integers occurring in the partition* π *of* n*. Show that* $\sum_{\pi \in p(n)} S(\pi) = \#p(n - 2)$*.*

Problem 6.24. $(*)$ *Prove that the number of partitions of* n *is equal to the number of partitions of* $2n$ *in precisely* n *parts.*

Problem 6.25. $(*)$ *Prove that the number of distinct ways to write* n *as the sum of distinct positive integers is equal to the number of ways to write* n *as the sum of positive odd integers.*

Problem 6.26. $(*)$ *Let* $p(n)$ *denote the number of partitions of* n. *Prove that for* $n \geqslant 2$, *we have that* $p(n) \geqslant 2p(n-1) + p(n-2)$.

Problem 6.27. $(*)$ *For any positive integer* n, *let* $p(n)$ *denote the set of distinct ways to write* n *as the sum of positive integers. For each partition* $\pi \in p(n)$, *let* $f(\pi)$ *denote the number of ones occurring in the partition* π *of* n *and* $g(p)$ *the number of distinct integers. Show that* $\sum_{\pi \in p(n)} f(\pi) = \sum_{\pi \in p(n)} g(\pi)$.

Problem 6.28. $(*)$ *A* perfect partition *of some* $n \in \mathbb{N}$ *is a sequence* $a_1 \geqslant \ldots \geqslant a_k$ *such that* $\sum_i a_i = n$ *and such that every* $1 \leqslant m \leqslant n$ *can be written in a unique way as the sum of some of the* a_i. *If two such* a_i *are equal, we regard them as indistinguishable. An* ordered factorisation *of some* $n \in \mathbb{N}$ *is an ordered tuple of positive integers, all distinct from 1, whose product equals* n. *Show that the number of perfect partitions of* n *equals the number of ordered factorisations of* $n+1$.

Problem 6.29. $(*)$ *An equilateral triangle of side length* n *is divided into* n^2 *small equilateral triangles of side length 1 in the most natural way. Determine the number of parallellograms you can draw on this grid.*

Problem 6.30. $(*)$ *For* $n \in \mathbb{N}$, *let* $\pi(n)$ *denote the number of sets of positive integers with the property that their sum is equal to* n. *Further, let* $\pi_2(n)$ *denote the number such sets containing at least one power of 2; here we regard 1 as a power of 2. Prove that* $\pi_2(n+1) = \pi(n)$.

Problem 6.31. $(*)$ *Let* a_n *denote the number of sequences of length* n *consisting of only zeroes and ones with the property that no three consecutive terms are equal to* $0, 1, 0$. *Let* b_n *be the number such sequences instead satisfying the property that no four consecutive terms are equal to either* $0, 0, 1, 1$ *or* $1, 1, 0, 0$ *respectively. Prove that* $b_{n+1} = 2a_n$.

Problem 6.32. $(*)$ *Let* n *be a positive integer. Every point* $(x, y) \in \mathbb{Z}^2_{\geqslant 0}$ *with* $x + y \leqslant n$ *is coloured either red or blue with the property that if* (x, y) *is red, then so are all of* (x', y') *with* $x' \leqslant x$ *and* $y' \leqslant y$. *Let* A *denote the number of ways to choose* n *blue points with distinct* x*-coordinates and let* B *denote the number of ways to choose* n *blue points with distinct* y *coordinates. Prove that* $A = B$.

Problem 6.33. $(*)$ *Let* $k \geqslant n$ *be positive integers with* $k + n$ *even. Consider an array of* $2n$ *lamps that are all initially turned off. A step consists of flicking the switch for precisely one lamp. Let* N *denote the number of sequences of* k *steps that result in the state in which the first* n *lamps are on and the latter* n *lamps are off. Let* M *be the number of sequences of* k *steps that result in a state in which the first* n *lamps are on and the latter* n *remained untouched throughout. Determine the value of* N/M.

6.5 IMC Problems

Problem 6.34. (1998) *Determine the number of functions* $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ *such that* $f(k) \leqslant f(k+1)$ *and* $f(k) = f(f(k+1))$ *for all* $1 \leqslant k \leqslant n-1$.

Problem 6.35. (1999) *We throw a fair dice* n *times. What is the probability that the sum of the values is divisible by* 5*?*

Problem 6.36. (2002) *Let* n *be a positive integer and define* $a_k = 1/\binom{n}{k}$ *and* $b_k = 2^{k-n}$ *for all* $1 \leqslant k \leqslant n$. *Prove that*

$$\frac{a_1 - b_1}{1} + \frac{a_2 - b_2}{2} + \ldots + \frac{a_n - b_n}{n} = 0.$$

Problem 6.37. (2002) *For each* $n \geqslant 1$, *let*

$$a_n = \sum_{k=0}^{\infty} \frac{k^n}{k!} \quad \text{and} \quad b_n = \sum_{k=0}^{\infty} (-1)^k \frac{k^n}{k!}.$$

Prove that $a_n \cdot b_n$ *is an integer.*

Problem 6.38. (2005) *For any integer* $n \geqslant 3$, *let* S_n *denote the set of* n*-tuples of numbers each from the set* $\{0, 1, 2\}$. *Let* A_n *be the subset of* S_n *containing all tuples without three consecutive equal terms. Let* B_n *be the subset of* S_n *containing all tuples without two consecutive zeroes. Prove that* $|A_{n+1}| = 3|B_n|$.

Problem 6.39. (2013)($*$) *Consider a circular necklace with 2013 beads. Each bead can be painted either white or green. A painting of the necklace is called good, if among any 21 successive beads there is at least one green bead. Prove that the number of good paintings of the necklace is odd. Here, two paintings that differ on some beads, but can be obtained from each other by rotating or flipping the necklace, are counted as different paintings.*

Problem 6.40. (2014)($*$) *Let* $A_1A_2 \ldots A_{3n}$ *be a closed broken line consisting of* $3n$ *line segments in the Euclidean plane. Suppose that no three of its vertices are colinear and that for each* $1 \leqslant i \leqslant 3n$, *the triangle* $A_iA_{i+1}A_{i+2}$ *has counter-clockwise orientation and* $\angle A_iA_{i+1}A_{i+2} = 60°$, *using the notation* $A_{3n+1} = A_1$ *and* $A_{3n+2} = A_2$. *Prove that the number of self-intersections of the broken line is at most* $3n^2/2 - 2n + 1$.

Problem 6.41. (2015) *Consider all* 26^{26} *words of length 26 in the Latin alphabet. Define the* weight *of a word as* $1/(k+1)$, *where* k *is the number of letters not used in the word. Prove that the sum of the weights of all words is* 3^{75}.

Problem 6.42. (2016) *Let* k *be a positive integer. For each integer* $n \geqslant 0$, *we let* $f(n)$ *be the number of solutions* $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ *to the inequality* $|x_1| + \ldots + |x_k| \leqslant n$. *Prove that for every* $n \geqslant 1$, *we have* $f(n-1)f(n+1) \leqslant f(n)^2$.

Problem 6.43. (2018) *Let* $\Omega = \{(x, y, z) \in \mathbb{Z}^3 \mid y + 1 \geqslant x \geqslant y \geqslant z \geqslant 0\}$. *A frog moves along the points of* Ω *by jumps of length 1. For every positive integer* n, *determine the number of paths the frog can take to reach* (n, n, n) *starting from* $(0, 0, 0)$ *in exactly* $3n$ *jumps.*

Problem 6.44. (2019) *Let* $x_1, \dots, x_n$ *be real numbers and write* $S = \{1, 2, \dots, n\}$. *For any* $I \subset S$, *let* $s(I) = \sum_{i \in I} x_i$. *Assume that the function* $I \mapsto s(I)$ *takes on at least* 1.8^n *values as* I *runs over all subsets of* S. *Prove that the number of* $I \subset S$ *for which* $s(I) = 2019$ *does not exceed* 1.7^n.

Problem 6.45. (2020) *Let* n *be a positive integer. Compute the number of words* w *of length* n *made up from the alphabet* $\{a, b, c, d\}$ *such that* w *contains an even number of* a*'s and an even number of* b*'s.*

Problem 6.46. (2021) *Let* n *and* k *be fixed positive integers and let* $a \geqslant 0$ *be an integer. Choose a* k*-element subset* X *of* $\{1, 2, \dots, k + a\}$ *uniformly at random and independently, choose a random* n*-element subset* Y *of* $\{1, 2, \dots, k + n + a\}$. *Prove that the probability of the event that* $\min(Y) > \max(X)$ *is independent from the value of* a.

Problem 6.47. (2022) *Let* p *be a prime number. A flea is staying at point 0 of the real line. At each minute, the flea has three possibilities: to stay at its position, or to move by 1 to the left or to the right. After* $p - 1$ *minutes, it wants to be at 0 again. Denote by* $f(p)$ *the number of its strategies to do this. Find* $f(p) \mod p$.

Problem 6.48. (2022)($*$) *Let* $n, k \geqslant 3$ *be integers and let* S *be a circle. Let* n *blue points and* k *red points be chosen uniformly and independently at random on the circle* S. *Denote by* F *the intersection of the convex hull of the red points and the convex hull of the blue points. Let* m *be the number of vertices of the convex polygon* F; *we set* $m = 0$ *when* F *is empty. Find the expected value of* m.

Problem 6.49. (2023)($*$) *Fix positive integers* n *and* k *such that* $2 \leqslant k \leqslant n$ *and a set* M *consisting of* n *fruits. A* permutation *is a sequence* $x = (x_1, x_2, \dots, x_n)$ *such that* $\{x_1, \dots, x_n\} = M$. *Ivan* prefers *some (at least one) of these permutations. He realised that for every preferred permutation* x, *there exist* k *indices* $i_1 < i_2 < \dots < i_k$ *with the following property: for every* $1 \leqslant j < k$, *if he swaps* x_{i_j} *and* $x_{i_{j+1}}$, *he obtains another preferred permutation. Prove that he prefers at least* $k!$ *permutations.*

Problem 6.50. (2023)($*$) *Let* T *be a tree with* n *verticies; that is, a connected simple graph on* n *vertices that contains no cycle. For every pair* u, v *of vertices, let* $d(u, v)$ *denote the distance between* u *and* v; *that is, the number of edges in the shortest path in* T *that connects* u *with* v. *Consider the sums*

$$W(T) = \sum_{\substack{\{u,v\} \subset V(T) \\ u \neq v}} d(u, v) \quad \textit{and} \quad H(T) = \sum_{\substack{\{u,v\} \subset V(T) \\ u \neq v}} \frac{1}{d(u, v)}.$$

Prove that

$$W(T) \cdot H(T) \geqslant \frac{(n-1)^3(n+2)}{4}.$$

7 Analysis III

We continue our treatment of real analysis by shifting our focus away from individual sequences and studying various kinds of functions instead. Many IMC problems deal with continuous functions, differentiable functions, integrals and inequalities; we lay out the most important results here before moving on to another great many exercises.

7.1 Continous functions

The reader should remember the following from their first course on real analysis.

Proposition 7.1. (Intermediate value theorem) *Let $f : U \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. If U is connected, then so is $f(U)$.*

Proposition 7.2. (Weierstraß's Theorem) *Let $f : C \to \mathbb{R}$ be a continuous function where C is compact. Then f attains both a maximum and a minimum on C.*

Finally, sometimes one needs a stronger version of continuity to draw the necessary conclusions.

Definition 7.3. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *uniformly continuous* if for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x, y \in \mathbb{R}$, we have that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Sometimes, we get uniform continuity for free.

Proposition 7.4. *Let $A, B \subset \mathbb{R}$ and let $f : A \to B$ be a continuous function. Suppose that A is compact. Then f is uniformly continuous.*

Continuous functions are always integrable, which is a very useful property to keep in mind. However, not every integrable function must necessarily be continuous. We record here a useful result about swapping the order of integration for multivariate functions that is occassionaly useful.

Proposition 7.5. (Fubini) *Given two intervals $I, J \subset \mathbb{R}$ and $f : I \times J \to \mathbb{R}$ such that*

$$\int_I \int_J |f(x,y)| dy dx < \infty,$$

it is true that

$$\int_I \int_J f(x,y) dy dx = \int_J \int_I f(x,y) dx dy.$$

7.2 Differentiable functions

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called *differentiable* at some $x \in \mathbb{R}$ if the limit

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists and is finite. Derivatives can be used in myriad ways to analyse real analysis problems. The following lemma should be known from high school.

Lemma 7.6. *Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. Suppose that f attains a local maximum or minimum at some $x \in (a, b)$. Then $f'(x) = 0$.*

It is very important to remember that the derivative of a continuous function need not be continuous itself. However, it still satisfies a sometimes very useful version of the intermediate value theorem.

Proposition 7.7. (Darboux's Theorem) *Let $A \subset \mathbb{R}$ be open and let $f : A \to \mathbb{R}$ be differentiable on A. For any interval $[a, b] \subset A$ and any $y \in \mathbb{R}$ satisfying $f'(a) < y < f'(b)$, there must exist some $x \in (a, b)$ such that $f'(x) = y$.*

Proposition 7.8. (Rolle's Theorem) *Let $f : [a, b] \to \mathbb{R}$ be a continuous function that is differentiable on (a, b), Then there exists some $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In particular, if $f(b) = f(a)$, then we can find some $c \in (a, b)$ such that $f'(c) = 0$. By choosing a clever function f to apply this result to, it can be used to prove the existence of some numbers satisfying rather spectacular equations, a few examples of which can be found in the exercises. Another theorem that can sometimes be used to the same effect, is the following.

Theorem 7.9. (Taylor's Theorem) *Let $k \in \mathbb{N}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function that is $k+1$ times differentiable. For any $a, x \in \mathbb{R}$, there exists some c in between a and x such that*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}.$$

A useful tool for computing limits is the following.

Proposition 7.10. (l'Hôpital) *Let f and g be differentiable functions on some subset $S \subset \mathbb{R}$ such that the limit*

$$L = \lim_{x\to s} \frac{f'(x)}{g'(x)}$$

exists for some $s \in S \cap \{\pm\infty\}$. Then, if either $\lim_{x\to s} f(x) = \lim_{x\to s} g(x) = 0$, or alternatively if $\lim_{x\to s} |g(x)| = \infty$, then also

$$\lim_{x\to s} \frac{f(x)}{g(x)} = L.$$

7.3 Limits and functions

The following theorem should be familiar.

Theorem 7.11. (Fundamental Theorem of Calculus) *Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function. Define the function* $F : [a, b] \to \mathbb{R}$ *by*

$$F(x) = \int_a^x f(t)dt.$$

Then F *is continuous on* $[a, b]$ *and differentiable on* (a, b), *satisfying* $F'(x) = f(x)$.

We note that F' and f need no longer agree if f was not originally assumed to be continuous. However, even in that case, F will still be continuous.

Some exercises concern not just limits of sequences, but limits of functions. There are various kinds of convergence that a sequence of functions can satisfy.

Definition 7.12. Let $A \subset \mathbb{R}$ and let $f_n : A \to \mathbb{R}$ be a function for every $n \in \mathbb{N}$. We say that this sequence converges *pointwise* to a function $f : A \to \mathbb{R}$ if for each $x \in A$, we have

$$\lim_{n\to\infty} f_n(x) = f(x).$$

We say that (f_n) converges *uniformly* to f if for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $n \geqslant N$ and all $x \in A$, it holds that $|f_n(x) - f(x)| < \epsilon$.

The following proposition shows why the notion of uniform convergence is useful and often preferred over pointwise convergence.

Proposition 7.13. *Let* (f_n) *be a sequence of functions* $A \to \mathbb{R}$ *for some subset* $A \subset \mathbb{R}$. *If all functions* f_n *are continuous and the sequence* (f_n) *converges uniformly to a function* $f : A \to \mathbb{R}$, *then* f *must also be continuous.*

It is important to not think of continuous functions as prettier than they are. There are some very exotic and nasty examples of continuous functions, like the "Devil's Staircase". For more such examples, we refer to "Counterexamples in Analysis" by B. R. Gelbaum and J. M. H. Olmsted. Finally, we have the following.

Theorem 7.14. (Dominated Convergence) *Let functions* $f_n(x)$ *for* $n \geqslant 1$ *and* $g(x)$ *be given on* $[a, b]$ *such that* $|f_n(x)| \leqslant g(x)$ *for all* n *and all* $x \in [a, b]$. *Suppose that*

$$\int_a^b g(x)dx < \infty$$

and that the pointwise limit $\lim_{n\to\infty} f_n(x)$ *exists for every* $x \in [a, b]$. *Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx.$$

7.4 Examples

Example 7.15. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that $f(f(x)) = x$ for some $x \in \mathbb{R}$. Prove that $f(y) = y$ for some $y \in \mathbb{R}$.

Solution: Suppose the contrary, so $f(y) \neq y$ for all $y \in \mathbb{R}$. Because f is continuous, this means that either $f(y) > y$ for all $y \in \mathbb{R}$, or $f(y) < y$ for all $y \in \mathbb{R}$. In the former case, one deduces that also $x = f(f(x)) > f(x) > x$; a contradiction. In the latter case, we similarly obtain a contradiction. △

Example 7.16. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function that is differentiable on $(0, 1)$, satisfying $f(0) = 0 = f(1)$. Prove that for any $\alpha \in \mathbb{R}$, there exists some $c \in (0, 1)$ such that

$$f'(c) + \alpha f(c) = 0.$$

Solution: Let $g(x) = e^{\alpha x} f(x)$. Then we also have $g(0) = 0 = g(1)$, so we may apply Rolle's theorem to g to find that for some $c \in (0, 1)$, it must hold that

$$g'(c) = 0 \iff e^{\alpha c} f'(c) + \alpha e^{\alpha c} f(c) = 0 \iff f'(c) + \alpha f(c) = 0;$$

this immediately completes the proof. △

7.5 Exercises

Problem 7.1. *Prove that any continuous function $f : [0, 1] \to \mathbb{R}$ with at least one zero must have a smallest and a biggest zero.*

Problem 7.2. *Let $a < b$ be real numbers and consider a continous function $f : [a, b] \to [a, b]$. Show that there exists some $c \in [a, b]$ such that $f(c) = c$.*

Problem 7.3. *Let $a_0, \ldots, a_n$ be real numbers satisfying the equation*

$$\frac{a_0}{1} + \frac{a_1}{2} + \ldots + \frac{a_n}{n+1} = 0.$$

Show that the polynomial $p(x) = a_n x^n + \ldots + a_1 x + a_0$ must have a real root.

Problem 7.4. *Find two non-constant continuously differentiable functions $f, g : [-1, 1] \to \mathbb{R}$ satisfying $(f + g)^2 = f^2 + g^2$. Show that two such functions also satisfy $(f \cdot g)' = f' \cdot g'$. Can you find functions further satisfying*

$$\int_0^x f(t)dt \cdot \int_0^x g(t)dt = \int_0^x f(t)g(t)dt \quad \textit{for all } x \in [0, 1]?$$

Problem 7.5. *Let $f : [0, 1] \to \mathbb{R}$ be a continuous function that is differentiable on $(0, 1)$, satisfying $f(0) = 0 = f(1)$. Prove that for any $\alpha \in \mathbb{R}$, there exists some $c \in (0, 1)$ such that*

$$f'(c) + \alpha f(c) \tan(c) = 0.$$

Problem 7.6. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a differentiable function. Prove that for some* $x \in \mathbb{R} \setminus \{0, 1\}$*, it holds that*
$$f'(x) = \left(\frac{2}{1-x} - \frac{2}{x}\right) f(x).$$

Problem 7.7. *Let* $f : [0, 1] \to [0, 1]$ *be a continuous function such that* $f(f(f(x))) = x$ *for all* $x \in [0, 1]$*. Prove that* $f(x) = x$ *for all* $x \in [0, 1]$.

Problem 7.8. *Let* $f : [0, 1] \to [0, 1]$ *be a differentiable function such that* $|f'(x)| \neq 1$ *for all* $x \in [0, 1]$*. Prove that there exist unique* $\alpha, \beta \in [0, 1]$ *such that* $f(\alpha) = \alpha$ *and* $f(\beta) = 1 - \beta$.

Problem 7.9. *Let* $f : [0, 1] \to \mathbb{R}$ *be a continuous function with* $f(0) = f(1)$*. Show that for every* $n \in \mathbb{N}$*, there exists some* $x \in [0, 1 - 1/n]$ *such that* $f(x) = f(x + 1/n)$.

Problem 7.10. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function with the properties that*
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(x) = 0 \quad \textit{and} \quad \lim_{x\to\infty} \frac{f''(x)}{f'(x)} = 2.$$
Determine $\lim_{x\to\infty} f'(x)/f(x)$.

Problem 7.11. *Let* $f(x) = e^{x^2}$*. Find an interval* (a, b) *and a function* $g : (a, b) \to \mathbb{R}$ *such that* $(f \cdot g)' = f' \cdot g'$ *on* (a, b).

Problem 7.12. *Let* $f : (a, b) \to \mathbb{R}$ *be a bounded convex function, i.e. for any* $x, y \in (a, b)$*, it holds that*
$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x) + f(y)}{2}.$$
Show that f *is continuous.*

Problem 7.13. *Let* $\alpha > 0$ *and let* $f : [0, 1] \to \mathbb{R}$ *be a differentiable function with the properties that* $f(0) = 0$ *and* $f(x) > 0$ *for all* $x > 0$*. Show that for some* $c > 0$*, it holds that*
$$\alpha \frac{f'(c)}{f(c)} = \frac{f'(1-c)}{f(1-c)}.$$

Problem 7.14. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a three times continuously differentiable function. Prove that for some* $a \in \mathbb{R}$,
$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geqslant 0.$$

Problem 7.15. *Let* $f : [-1, 1] \to \mathbb{R}$ *be a twice differentiable function with* $f(-1) = f(0) = f(1)$*. Prove that for some* $x \in (-1, 1)$*, it holds that* $f''(x) = f'(x)^2$.

Problem 7.16. *For which* $\alpha \in \mathbb{R}$ *can we find some continuous function* $f : \mathbb{R} \to \mathbb{R}$ *such that* $f(f(x)) = \alpha x^9$*?*

Problem 7.17. *Let* $f, g : \mathbb{R} \to \mathbb{R}$ *be non-constant differentiable functions satisfying for any* $x, y \in \mathbb{R}$ *the equations*
$$f(x+y) = f(x)f(y) - g(x)g(y) \quad \textit{and} \quad g(x+y) = f(x)g(y) + g(x)f(y).$$
If $f'(0) = 0$*, show that* $f(x)^2 + g(x)^2 = 1$ *for all* $x \in \mathbb{R}$.

Problem 7.18. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that for any integer* $k \geqslant 1$, *it holds that*

$$\int_0^k f(x)^2 dx = \int_0^k f(x)f(k-x)dx.$$

Prove that $f(2023) = f(2024)$.

Problem 7.19. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function. We define* $g : \mathbb{R} \to \mathbb{R}$ *by*

$$g(x) = f(x)\int_0^x f(t)dt.$$

Suppose that g *is non-decreasing. Prove that* $f = 0$.

Problem 7.20. $(*)$ *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a twice differentiable function satisfying*

$$f(x) + f''(x) = -xg(x)f'(x)$$

for some function $g : \mathbb{R} \to \mathbb{R}$ *satisfying* $g(x) \geqslant 0$ *for all* $x \in \mathbb{R}$. *Prove that* $|f(x)|$ *is bounded.*

Problem 7.21. $(*)$ *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a three-times differentiable function with at least five zeroes. Prove that* $f + 6f' + 12f'' + 8f'''$ *has at least two zeroes.*

Problem 7.22. $(*)$ *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a three times continuously differentiable function satisfying that all of* f, f', f'' *and* f''' *are positive on all of* $\mathbb{R}$. *Suppose further that* $f'''(x) \leqslant f(x)$ *for all* $x \in \mathbb{R}$. *Prove that* $f'(x) < 2f(x)$ *for all* $x \in \mathbb{R}$.

7.6 IMC Problems

Problem 7.23. (1994) *Let* $b > 0$ *and let* $f : [0, b] \to \mathbb{R}$ *be continuous. If* $g : \mathbb{R} \to \mathbb{R}$ *is a periodic function with period* b, *prove that*

$$\lim_{n\to\infty}\int_0^b f(x)g(nx)dx = \frac{1}{b}\int_0^b f(x)dx \cdot \int_0^b g(x)dx.$$

Problem 7.24. (1994)$(*)$ *Let* $N > 0$ *and let* $f : [0, N] \to \mathbb{R}$ *be a twice continuously differentiable function with the property that* $|f'(x)| < 1$ *and* $f''(x) > 0$ *for all* $x \in [0, N]$. *Suppose that* $0 \leqslant m_0 < m_1 < \ldots < m_k \leqslant N$ *are integers such that* $n_i = f(m_i)$ *are also integers. Denote* $b_i = n_i - n_{i-1}$ *and* $a_i = m_i - m_{i-1}$. *Prove that*

$$-1 < \frac{b_1}{a_1} < \ldots < \frac{b_k}{a_k} < 1.$$

Also prove that for every $A > 1$, *there are no more than* N/A *indices* $j \in \{0, \ldots, k\}$ *such that* $a_j > A$. *Conclude that* $k \leqslant 3N^{2/3}$.

Problem 7.25. (1994) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a smooth function and let* $a, b \in \mathbb{R}$ *be such that*

$$\log\left(\frac{f(b) + f'(b) + \ldots + f^{(n)}(b)}{f(a) + f'(a) + \ldots + f^{(n)}(a)}\right) = b - a.$$

Show that for some $c \in (a, b)$, *it holds that* $f^{(n+1)}(c) = f(c)$.

Problem 7.26. (1995) *Let* $f : (0, \infty) \to \mathbb{R}$ *be twice continuously differentiable. Suppose that* $\lim_{x \to 0} f'(x) = -\infty$ *and* $\lim_{x \to 0} f''(x) = \infty$. *Show that* $\lim_{x \to 0} f(x)/f'(x) = 0$.

Problem 7.27. (1995) *Let* $F : (1, \infty) \to \mathbb{R}$ *be the function defined by*

$$F(x) = \int_x^{x^2} \frac{dt}{\log(t)}.$$

Show that F *is injective and determine its range.*

Problem 7.28. (1995) *Prove that every function of the form*

$$f(x) = \frac{a_0}{2} + \cos(x) + \sum_{n=2}^{N} a_n \cos(nx)$$

with $|a_0| < 1$ *has positive as well as negative values in* $[0, 2\pi)$.

Problem 7.29. (1995) *Suppose that we are given an infinite sequence of continuous functions* $f_n : [0, 1] \to \mathbb{R}$ *for each* $n \geqslant 1$ *such that*

$$\int_0^1 f_m(x) f_n(x) dx = \begin{cases} 1 & \textit{if } m = n; \\ 0 & \textit{otherwise.} \end{cases}$$

Suppose further that

$$\sup\{|f_n(x)| : x \in [0, 1], n \in \mathbb{N}\} < \infty.$$

Show that $\lim_{k \to \infty} f_k(x)$ *does not exist for all* $x \in [0, 1]$.

Problem 7.30. (1996) *Let* $f : [0, 1] \to [0, 1]$ *be a continuous function. Consider any sequence given by* $x_{n+1} = f(x_n)$. *Show that* (x_n) *converges if and only if* $\lim_{n \to \infty}(x_{n+1} - x_n) = 0$.

Problem 7.31. (1997) *Let* f *be a three times continuously differentiable function with the property that* $f(0) = f'(0) = 0 < f''(0)$. *Define*

$$g(x) = \frac{d}{dx} \left(\frac{\sqrt{f(x)}}{f'(x)} \right)$$

for $x \neq 0$ *and* $g(0) = 0$. *Show that* g *is bounded in a neighbourhood of* 0. *Does this conclusion still follow if* f *is only twice continuously differentiable?*

Problem 7.32. (1998) *Let* $f : [0, 1] \to \mathbb{R}$ *be given by* $f(x) = 2x(1 - x)$. *Denote* $f_n = f \circ \ldots \circ f$ *for the* n*-fold composition of* f. *Determine*

$$\int_0^1 f_n(x) dx.$$

Does this sequence converge?

Problem 7.33. (1998) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a twice differentiable function satisfying* $f(0) = 2$, $f'(0) = -2$ *and* $f(1) = 1$. *Prove that there exists some* $c \in (0, 1)$ *such that*

$$f(c) \cdot f'(c) + f''(c) = 0.$$

Problem 7.34. (1998) *Let* $c \in (0, 1)$ *and define*

$$f(x) = \begin{cases} x/c & \text{if } x \in [0, c]; \\ (1-x)/(1-c) & \text{if } x \in [c, 1]. \end{cases}$$

Show that the equation $f^n(x) = x$ *only has finitely many solutions for every* $n \geqslant 1$, *where* f^n *denotes the* n*-fold composition of* f *with itself. Show also that for every* n, *there are solutions to* $f^n(x) = x$ *for which this* n *is minimal.*

Problem 7.35. (1998)($*$) *Let* $f : (0, 1) \to [0, \infty)$ *be a function that is non-zero only on the distinct points* $a_1, a_2, \ldots$. *Suppose that*

$$\sum_{n=1}^{\infty} f(a_n) < \infty.$$

Prove that f *is differentiable at at least one* $x \in (0, 1)$.

Problem 7.36. (2000) *Let* $f : [0, 1] \to [0, 1]$ *be a strictly increasing function. Show that for some* $x \in [0, 1]$, *it holds that* $f(x) = x$. *What if* f *is strictly decreasing?*

Problem 7.37. (2002) *Let* $f : [a, b] \to [a, b]$ *be a continuous function and let* $p \in [a, b]$. *Define the sequence* (p_n) *by* $p_0 = p$ *and* $p_{n+1} = f(p_n)$ *for all* $n \geqslant 0$. *Suppose that the set* $T_p = \{p_n \mid n \in \mathbb{N}\}$ *is closed. Prove that* T_p *is finite.*

Problem 7.38. (2003)($*$) *Let* $g : [0, 1] \to \mathbb{R}$ *be a function and define the sequence of functions* $f_n : (0, 1] \to \mathbb{R}$ *by setting* $f_0(x) = g(x)$ *and*

$$f_{n+1}(x) = \frac{1}{x} \int_0^x f_n(t)\,dt$$

for all $x \in (0, 1]$ *and* $n \geqslant 0$. *Determine* $\lim_{n\to\infty} f_n(x)$ *for every* $x \in (0, 1]$.

Problem 7.39. (2004) *Let* $f, g : [a, b] \to [0, \infty)$ *be continuous and non-decreasing functions such that for each* $x \in [a, b]$, *we have*

$$\int_a^x \sqrt{f(t)}\,dt \leqslant \int_a^x \sqrt{g(t)}\,dt,$$

with equality for $x = b$. *Prove that*

$$\int_a^b \sqrt{1 + f(t)}\,dt \geqslant \int_a^b \sqrt{1 + g(t)}\,dt.$$

Problem 7.40. (2005) *Let* $f(x) = x^2 + bx + c$ *where* $b, c \in \mathbb{R}$. *Now define the possibly empty set* $M = \{x \in \mathbb{R} : |f(x)| < 1\}$. *Clearly* M *is the union of disjoint open intervals. Let* $|M|$ *denote the sum of their lenghts. Prove that* $|M| \leqslant 2\sqrt{2}$.

Problem 7.41. (2005) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a three times differentiable function. Prove that for some* $c \in (-1, 1)$, *it holds that*

$$f'''(c) = 3f(1) - 3f(-1) - 6f'(0).$$

Problem 7.42. (2006) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a surjective monotonous function. Show that* f *must be continuous.*

Problem 7.43. (2006) *Determine all functions* $f : \mathbb{R} \to \mathbb{R}$ *such that for any real numbers* $a < b$, *the image* $f([a, b])$ *is a closed interval of length* $b - a$.

Problem 7.44. (2007) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function with the property that for every* $c > 0$, *the graph of* f *can be transformed into the graph of* $c \cdot f$ *by translating and rotating. Does this imply that* $f(x) = ax + b$ *for some* $a, b \in \mathbb{R}$?

Problem 7.45. (2007) *Let* $C \subset \mathbb{R}$ *be a compact set and let* $f : C \to C$ *be a non-decreasing continuous function. Show that for some* $p \in C$ *it holds that* $f(p) = p$.

Problem 7.46. (2008) *Determine all continuous functions* $f : \mathbb{R} \to \mathbb{R}$ *with the property that*

$$x - y \in \mathbb{Q} \implies f(x) - f(y) \in \mathbb{Q}.$$

Problem 7.47. (2009) *Let* $f, g : \mathbb{R} \to \mathbb{R}$ *be two functions with the property that* $f(r) \leqslant g(r)$ *for all* $r \in \mathbb{Q}$. *Is* f *and* g *are non-decreasing, does this imply that* $f(x) \leqslant g(x)$ *for all* $x \in \mathbb{R}$? *What if* f *and* g *are both continuous?*

Problem 7.48. (2011) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function. We say some* $x \in \mathbb{R}$ *is a shadow-point of* f *if for some* $y > x$ *it holds that* $f(y) > f(x)$. *Let* $a < b$ *now be two real numbers that are not shadow-points of* f, *with the property that all points in* (a, b) *are shadow-points of* f. *Show that* $f(x) \leqslant f(b)$ *for all* $x \in (a, b)$ *and that* $f(a) = f(b)$.

Problem 7.49. (2015) *Compute*

$$\lim_{A \to \infty} \frac{1}{A} \int_1^A A^{1/x} dx.$$

Problem 7.50. (2016) *Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function that is differentiable on* (a, b). *Suppose that* f *has infinitely many zeroes, but that there is no* $x \in (a, b)$ *such that* $f(x) = f'(x) = 0$. *Prove that* $f(a)f(b) = 0$ *and give an example of such a function on* $[0, 1]$.

Problem 7.51. (2017) *Let* $f : [0, \infty) \to \mathbb{R}$ *be a continuous function satisfying the property that* $\lim_{x \to \infty} f(x) = L$ *exists. Prove that*

$$\lim_{n \to \infty} \int_0^1 f(nx) dx = L.$$

Problem 7.52. (2017)($*$) *Define the sequence* $f_1, f_2, \ldots : [0,1] \to \mathbb{R}$ *of continuously differentiable functions by*

$$f_1 = 1, \quad f'_{n+1} = f_n f_{n+1} \text{ on } (0,1) \quad \text{and} \quad f_{n+1}(0) = 1.$$

Show that $\lim_{n\to\infty} f_n(x)$ *exists for every* $x \in [0,1)$ *and determine the limit function.*

Problem 7.53. (2019) *Let* $f, g : \mathbb{R} \to \mathbb{R}$ *be continuous functions such that* g *is differentiable. Suppose that*

$$\big(f(0) - g'(0)\big)\big(g'(1) - f(1)\big) > 0.$$

Show that there exists some $c \in (0,1)$ *such that* $f(c) = g'(c)$.

Problem 7.54. (2023) *Let* V *be the set of all continuous functions* $f : [0,1] \to \mathbb{R}$, *differentiable on* $(0,1)$, *with the property that* $f(0) = 0$ *and* $f(1) = 1$. *Determine all* $\alpha \in \mathbb{R}$ *such that for every* $f \in V$, *there exists some* $\xi \in (0,1)$ *such that* $f(\xi) + \alpha = f'(\xi)$.

8 Algebra III

Polynomials often make for interesting problems because by their very nature, they exist on the crossroads of algebra and analysis; those who enjoy algebraic geometry might also add this field into the mix. In many problems about polynomials, one requires ideas from at least two of these fields; a dichotomy which is reflected in both the introductory texts and the problems themselves.

8.1 Basics on polynomials

A monomial is a term of the form ax^n with $n \geqslant 0$ and typically $a \in k$ with $k \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. A polynomial is a finite linear combination of monomials. All non-zero polynomials have a *degree*, which is defined as the largest power of x whose coefficient is non-zero; one typically writes $\deg(0) = -\infty$. We can do division with remainder.

Lemma 8.1. **??** *Let* k *be a field and let* $p(x), g(x) \in k[x]$*. Then there exist unique polynomials* $q(x), r(x) \in k[x]$ *with* $\deg(r) < \deg(g)$ *such that* $p(x) = q(x)g(x) + r(x)$*. In particular, we see that* $x - a$ *is a divisor of* $p(x) - p(a)$ *for any* $a \in k$*.*

The above can be used to show that the ring $k[x]$ is always a principal ideal domain. By the above, the number of zeroes of a non-zero polynomial is bounded from above by its degree. If $p(x) \in \mathbb{R}[x]$ and $\alpha \in \mathbb{C}$ is a zero of p, then so is $\overline{\alpha}$. As a result, a real polynomial of odd degree must have at least one real zero. The following is more specific to $\mathbb{C}$.

Theorem 8.2. (Fundamental Theorem of Algebra) *A polynomial in* $\mathbb{C}[X]$ *of degree* $n \geqslant 0$ *can be written as* $c(x - r_1) \cdots (x - r_n)$ *for certain* $r_1, \ldots, r_n \in \mathbb{C}$*, unique up to reordering.*

The number of times a certain complex number appears in the mutliset $\{r_1, \ldots, r_n\}$ is called the *multiplicity* of the zero at that point. Given the zeroes $r_1, \ldots, r_n$ of a poynomial, it is easy to reconstruct its coefficients using symmetric polynomials, which is illustrated by the example

$$(x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc.$$

It follows from all the above that a polynomial of degree n is uniquely determined by its values at $n + 1$ different points.

8.2 Examples

Example 8.3. Determine all polynomials P satisfying $P(0) = 1$ and $P(x^2+1) = P(x)^2+1$.

Solution: Define the sequence $a_0 = 0$ and $a_{n+1} = a_n^2 + 1$ for all $n \geqslant 0$. We claim that $P(a_n) = a_n^2 + 1$ for all $n \geqslant 0$. Indeed, $P(0)^2 = 0^2 + 1$ and if this holds for a given n, then we compute that

$$P(a_{n+1}) = P(a_n^2+1) = P(a_n)^2+1 = (a_n^2+1)^2+1 = a_{n+1}^2+1;$$

proving our claim by induction. This means that the polynomial $Q(x) = P(x) - x^2 - 1$ satisfies $Q(a_n) = 0$ for all $n \geqslant 0$. However, only the zero polynomial can have infinitely many zeroes. As such, $Q = 0$ and hence $P(x) = x^2 + 1$. One checks that this polynomial indeed satisfies the given equation tautologically, showing that it is the only one. △

Example 8.4. Determine all real polynomials $p(x)$ of degree $n \geqslant 0$ such that

$$(n-2)p'(x)p''(x) = np(x)p'''(x)$$

for all $x \in \mathbb{R}$.

Solution: If $n \leqslant 1$, then both sides of the equation vanish. If $n = 2$, then the same happens, as $n - 2 = 0$ and $p''' = 0$ in that case. We thus henceforth assume that $n \geqslant 3$.

Consider the functions $f(x) = (n-2)\log p(x)$ and $g(x) = n \log p''(x)$. We claim that $f'(x) = g'(x)$ for all $x \in \mathbb{R}$. Indeed, we compute that

$$f'(x) = (n-2)\frac{p'(x)}{p(x)} \quad \text{and} \quad g'(x) = n\frac{p'''(x)}{p''(x)}.$$

Therefore, we find that

$$f'(x) = g'(x) \iff (n-2)\frac{p'(x)}{p(x)} = n\frac{p'''(x)}{p''(x)} \iff (n-2)p'(x)p''(x) = np(x)p'''(x);$$

this is precisely our assumption. We conclude that $f(x) = g(x) + c$ for a certain fixed $c \in \mathbb{R}$. In other words,

$$f(x) - g(x) = c \iff (n-2)\log p(x) - n\log p''(x) = c \iff \log\left(\frac{p(x)^{n-2}}{p''(x)^n}\right) = c.$$

We claim that for some constant $C \in \mathbb{R}$, it must hold that $p(x)^{n-2} = Cp''(x)^n$ for all $x \in \mathbb{R}$. Let $\alpha \in \mathbb{C}$ be any root of either side of the equation with multiplicity d. Then the left hand side implies that $n - 2 \mid d$, whereas the right hand side yields that $n \mid d$.

If n is odd, these two relations imply that $n(n-2) \mid d$ and as $d \leqslant n(n-2)$ as this is the degree of the polynomial on both sides, equality must follow. This shows that $p(x) = a(x-b)^n$ for some $a, b \in \mathbb{R}$ and we leave it to the reader to check that such polynomials always satisfy the given equation.

If n is even, then using similar reasoning we find that d must be either $n(n-2)$ or $n(n-2)/2$. The former option is identical to the case treated above, so suppose we are in the latter. Then there must be two distinct roots of multiplicity $n(n-2)/2$, so that $p(x) = a(x-b)^{n/2}(x-\overline{b})^{n/2}$ for some $a \in \mathbb{R}$ and $b \in \mathbb{C}$. One checks again that such polynomials always satisfy the given equation, solving the problem. △

8.3 Exercises

Problem 8.1. *Prove that* $x^2 + x + 1$ *divides* $x^{2n} + x^n + 1$ *if and only if* n *is not divisible by 3.*

Problem 8.2. *Determine all polynomials* $p(x) \in \mathbb{R}[x]$ *satisfying* $p(x^2) = p(x) \cdot p(x+2)$.

Problem 8.3. *Let* a, b, c *be the roots of* $p(x) = 3x^3 - 14x^2 + x + 62$. *Determine*

$$\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3}.$$

Problem 8.4. *Let* $a_0, \ldots, a_n > 0$. *Prove that the polynomial*

$$p(x) = x^{n+1} - a_n x^n - \ldots - a_1 x - a_0$$

has a unique positive zero.

Problem 8.5. *Define the function* $f : \mathbb{N} \to \mathbb{N}$ *by* $f(n) = 1! + 2! + \ldots + n!$. *Determine polynomials* $p(x), q(x) \in \mathbb{Z}[x]$ *such that* $f(n+2) = p(n)f(n+1) + q(n)f(n)$ *for all* $n \in \mathbb{N}$.

Problem 8.6. *What is the smallest number of non-zero coefficients a degree 5 polynomial* $p(x) \in \mathbb{Z}[x]$ *can have such that all its zeroes are real?*

Problem 8.7. *Does there exist an infinite sequence of non-zero reals* $a_0, a_1, \ldots$ *such that for each* $n \geqslant 1$, *the polynomial* $p_n(x) = a_0 + a_1 x + \ldots + a_n x^n$ *has* n *distinct real zeroes?*

Problem 8.8. *Consider all lines that intersect the graph of* $p(x) = 2x^4 + 7x^3 + 3x - 5$ *in four distinct points with* x*-coordinates* x_1, x_2, x_3, x_4. *Prove that* $x_1 + x_2 + x_3 + x_4$ *is independent of the choice of such a line. What is its value?*

Problem 8.9. *Let* a, b, c *denote the three real zeroes of* $p(x) = 5x^3 + 4x^2 - 8x + 6$. *Determine* $a(1 + b + c) + b(1 + a + c) + c(1 + a + b)$.

Problem 8.10. *Does there exist a polynomial* $p(x) \in \mathbb{R}[x]$ *such that* $p(1/k) = (k+2)/k$ *for all positive integers* k? *What about* $p(k) = 1/(2k+1)$?

Problem 8.11. *Determine all triples of complex numbers* $x, y, z \in \mathbb{C}$ *satisfying*

$$\begin{cases} x + y + z = 3; \\ x^2 + y^2 + z^2 = 11; \\ x^3 + y^3 + z^3 = 27. \end{cases}$$

Problem 8.12. *Let* $a_1, \ldots a_n$ *and* $b_1, \ldots, b_n$ *be complex numbers. Consider the* $n \times n$*-array of numbers with the number* $a_i + b_j$ *in cell* (i, j) *for all* $1 \leqslant i, j \leqslant n$. *Suppose that the product of all numbers in any given row is independent from the choice of this row. Prove that the product of the numbers in any given column is also independent from the choice of the column.*

Problem 8.13. *Suppose that the product of two of the roots of the polynomial* $x^4 - 18x^3 + kx^2 + 200x - 1984$ *is equal to* -32. *Determine* k.

Problem 8.14. *In a tournament, n teams each play one game against every other team; no game can end in a draw. Let w_i be the number of wins of team i and let ℓ_i denote the number of losses. Prove that $\sum_{i=1}^n w_i^2 = \sum_{i=1}^n \ell_i^2$.*

Problem 8.15. *On a blackboard, some numbers are written. A move consists of choosing two of them, say a and b, and replacing them by $2ab - a - b + 1$. If we start with the numbers $49/1, 49/2, \ldots, 49/97$, what possibilities are there for the final number on the board? And what if we start with $1/2016, 2/2016, \ldots, 2015/2016$?*

Problem 8.16. *Prove that for all $|x| < 1$,*

$$\sum_{k=0}^{\infty} x^k \frac{1 + x^{2k+2}}{(1 - x^{2k+2})^2} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{(1 - x^{k+1})^2}.$$

Problem 8.17. *Let $p(x) = (x-1)(x-2)(x-3)$. How many $q \in \mathbb{R}[x]$ are there with the property that for some polynomial $r(x)$ of degree at most 3, it holds that $p(q(x)) = p(x) \cdot r(x)$?*

Problem 8.18. *Let $p(x) \in \mathbb{C}[x]$ be a polynomial with the property that $p'(x)$ has at least two distinct zeroes. Prove that*

$$\inf_{r \in \mathbb{R}} \max\{|z - w| : p(z) = p(w) = r\} > 0.$$

Problem 8.19. *Can we find polynomials $a(x)$, $b(x)$, $c(y)$ and $d(y)$ such that*

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)?$$

Problem 8.20. *Let $p(x) \in \mathbb{R}[x]$ be a polynomial with the property that $p(x) \geqslant 0$ for all $x \in \mathbb{R}$. Prove that there exist polynomials $f_1, \ldots, f_k \in \mathbb{R}[x]$ such that*

$$p(x) = \sum_{j=1}^{k} f_j(x)^2.$$

Problem 8.21. *Determine all polynomials $p(x) \in \mathbb{R}[x]$ of degree $n \geqslant 2$ with the property that there exists real $r_1 < \ldots < r_n$ with $p(r_i) = 0$ for all $1 \leqslant i \leqslant n$ and $p'((r_i + r_{i+1})/2) = 0$ for all $1 \leqslant i < n$.*

Problem 8.22. $(*)$ *For any polynomial $p(x) \in \mathbb{R}[x]$, let $\Gamma(p)$ denote the sum of the squares of its coefficients. Given $f(x) = 3x^2 + 7x + 2$, determine a polynomial $g(x)$ with $g(0) = 1$ and the property that $\Gamma(g^n) = \Gamma(f^n)$ for all $n \geqslant 1$.*

8.4 IMC Problems

Problem 8.23. (1994) *Let A and B be real $n \times n$ matrices. Assume that there exist $n + 1$ distinct real numbers $t_0, \ldots, t_n$ such that the matrices $A + t_iB$ are nilpotent for all $0 \leqslant i \leqslant n$. Prove that both A and B must be nilpotent themselves.*

Problem 8.24. (1995) *Let* $p(x) \in \mathbb{C}[x]$ *be a polynomial of degree* n *all of whose zeroes lie on the complex unit circle. Prove that all the roots of the polynomial* $2xp'(x) - np(x)$ *also lie on the unit circle.*

Problem 8.25. (1998) *Let* $p(x) \in \mathbb{R}[x]$ *be a polynomial of degree* n *with only real zeroes. Prove that for all* $x \in \mathbb{R}$, *it holds that* $(n-1)p'(x)^2 \geqslant np(x)p''(x)$. *When does equality hold?*

Problem 8.26. (1998) *Let* $\mathcal{P}$ *be the set of real polynomials of degree at most 3 with the property that* $|f(\pm 1)| \leqslant 1$ *and* $|f(\pm 1/2)| \leqslant 1$. *Determine* $\sup_{f \in \mathcal{P}} \max_{x \in [-1,1]} |f''(x)|$.

Problem 8.27. (2000) *Let* $p(x) = x^5 + x$ *and* $q(x) = x^5 + x^2$. *Determine all pairs of distinct complex numbers* $w, z \in \mathbb{C}$ *such that* $p(w) = p(z)$ *and* $q(w) = q(z)$.

Problem 8.28. (2000) *Let* p *be a complex polynomial of degree* $n \geqslant 1$. *Prove that there are at least* $n+1$ *distinct* $z \in \mathbb{C}$ *such that* $p(z) \in \{0, 1\}$.

Problem 8.29. (2001) *Let* k *be a positive integer and let* $p(x)$ *be a polynomial of degree* n, *each of whose coefficients is taken from the set* $\{0, \pm 1\}$, *which is also required to be divisible by* $(x-1)^k$. *Let* q *be a prime such that* $q\log(n+1) < k\log(q)$. *Prove that the complex* q*-th roots of unity are roots of* $p(x)$.

Problem 8.30. (2001) *Suppose that for some positive integer* k, *the polynomial* $1 + x + \ldots + x^k$ *is written as the product of two real polynomials with non-negative coefficients. Prove that all coefficients from both polynomials in the factorisation are taken from the set* $\{0, 1\}$.

Problem 8.31. (2003)($*$) *Let* $f(z) = a_n z^n + \ldots + a_1 z + a_0 \in \mathbb{R}[z]$. *Prove that if all roots* z *of* f *satisfy* $\Re(z) < 0$, *then* $a_k a_{k+3} < a_{k+1} a_{k+2}$ *for all* $0 \leqslant k \leqslant n-3$.

Problem 8.32. (2004) *Let* $p(x) = x^2 - 1$. *How many distinct real solutions does the equation* $(p \circ p \circ \ldots \circ p)(x) = 0$ *have, where we compose* p *with itself* 2004 *times?*

Problem 8.33. (2005) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function with the property that* $f(x)^n \in \mathbb{R}[x]$ *for all* $n \geqslant 2$. *Prove that* $f \in \mathbb{R}[x]$.

Problem 8.34. (2007) *Call a polynomial* $p(x_1, \ldots, x_k)$ good *if there exist real* 2×2*-matrices* $A_1, \ldots, A_k$ *such that*

$$p(x_1, \ldots, x_k) = \det\Big(\sum_{i=1}^{k} x_i A_i\Big).$$

Find all values of k *for which all homogeneous polynomials with* k *variables of degree 2 are good.*

Problem 8.35. (2007)($*$) *Let* $f(x) \in \mathbb{R}[x]$ *be a non-zero polynomial. Define the sequence* $f_0, f_1, \ldots$ *of polynomials by* $f_0 = f$ *and* $f_{n+1} = f_n + f_n'$ *for every* $n \geqslant 0$. *Prove that there exists some* $N \in \mathbb{N}$ *such that for ever* $n \geqslant N$, *all roots of* f_n *are real.*

Problem 8.36. (2008) *Let* V *be the real vector space of all real polynomials in one variable and let* $P : V \to \mathbb{R}$ *be a linear map. Suppose that for all* $f, g \in V$ *with* $P(fg) = 0$, *we have either* $P(f) = 0$ *or* $P(g) = 0$. *Prove that there exist* $x_0, c \in \mathbb{R}$ *such that* $P(f) = cf(x_0)$ *for all* $f \in V$.

Problem 8.37. (2008) *Let* $n, k \in \mathbb{N}$ *and suppose that the polynomial* $x^{2k} - x^k + 1$ *divides* $x^{2n} + x^n + 1$*. Prove that also* $x^{2k} + x^k + 1$ *divides* $x^{2n} + x^n + 1$*.*

Problem 8.38. (2009)($*$) *Let* $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ *be a complex polynomial. Suppose that* $1 = c_0 \geqslant c_1 \geqslant \ldots \geqslant c_n \geqslant 0$ *is a sequence of real numbers satisfying* $2c_k \leqslant c_{k-1} + c_{k+1}$ *for all* $1 \leqslant k \leqslant n-1$*. Define* $q(z) = c_0 a_0 + c_1 a_1 z + \ldots + c_n a_n z^n$*. Prove that* $\max_{|z|\leqslant 1} |q(z)| \leqslant \max_{|z|\leqslant 1} |p(z)|$*.*

Problem 8.39. (2010) *Suppose that for a function* $f : \mathbb{R} \to \mathbb{R}$ *and real numbers* $a < b$*, one has* $f(x) = 0$ *for all* $x \in (a, b)$*. Prove that* $f(x) = 0$ *for all* $x \in \mathbb{R}$ *if*

$$\sum_{k=0}^{p-1} f\left(y + \frac{k}{p}\right) = 0$$

for every prime number p *and every* $y \in \mathbb{R}$*.*

Problem 8.40. (2012) *Homer Simpson and Albert Einstein are playing a game. They take turns choosing a coefficient* a_i *of the polynomial* $p(x) = x^{2012} + a_{2011}x^{2011} + \ldots + a_1 x + a_0$ *and assigning it some real value. Once a coefficient has been assigned, it can never be changed again. Homer starts and it is his goal to ensure that* $p(x)$ *is divisible by some agreed upon polynomial* $m(x)$*. Who has the winning strategy for* $m(x) = x - 2012$*? And what about* $m(x) = x^2 + 1$*?*

Problem 8.41. (2014) *Let* $n \geqslant 1$ *be an integer. Prove that we can find* $a_0, \ldots, a_n > 0$ *such that for every choice of signs, the polynomial* $\pm a_n x^n \pm a_{n-1} x^{n-1} \pm \ldots \pm a_1 x \pm a_0$ *has precisely* n *distinct real zeroes.*

Problem 8.42. (2015)($*$) *Let* n *be a positive integer and let* $p(x) \in \mathbb{Z}[x]$ *be a polynomial of degree* n*. Prove that* $\max_{0\leqslant x\leqslant 1} |p(x)| > 1/e^n$*.*

Problem 8.43. (2017)($*$) *Let* $k, n \in \mathbb{N}$ *with* $n \geqslant k^2 - 3k + 3$*, and let* $f(z) = z^{n-1} + c_{n-2}z^{n-2} + \ldots + c_0$ *be a complex polynomial such that* $c_i c_j = 0$ *whenever* $i + j = n - 2$*. Prove that* $f(z)$ *and* $z^n - 1$ *has at most* $n - k$ *common roots.*

Problem 8.44. (2017)($*$) *Let* $p(x) \in \mathbb{R}[x]$ *be non-constant. For every* $n \in \mathbb{N}$*, define* $q_n(x) = (x+1)^n p(x) + x^n p(x+1)$*. Prove that there are only finitely many* n *for which all roots of* $q_n(x)$ *are real.*

Problem 8.45. (2019)($*$) *Determine all pairs* $p(x)$*,* $q(x)$ *of monic complex polynomials such that* $p(x)$ *divides* $q(x)^2 + 1$ *and* $q(x)$ *divides* $p(x)^2 + 1$*.*

Problem 8.46. (2020)($*$) *A polynomial* $p(x) \in \mathbb{R}[x]$ *satisfies the equation* $p(x+1) - p(x) = x^{100}$ *for all* $x \in \mathbb{R}$*. Prove that* $p(1-t) \geqslant p(t)$ *for all* $0 \leqslant t \leqslant 1/2$*.*

Problem 8.47. (2021) *Let* $D \subset \mathbb{C}$ *be an open set containing the closed unit disk* $\{z : |z| \leqslant 1\}$*. Let* $f : D \to \mathbb{C}$ *be a holomorphic function and let* $p(z)$ *be a monic polynomial. Prove that* $|f(0)| \leqslant \max_{|z|=1} |f(z)p(z)|$*.*

Problem 8.48. (2023) *Find all polynomials* P *in two variables with real coefficients satisfying the identity*

$$P(x, y)P(z, t) = P(xz - yt, xt + yz).$$

9 Linear Algebra II

It is very common at the IMC to heavily exploit the intricate and delicate logic associated with the interplay between matrices and polynomials of them. These are quite closely linked to the theory of eigenvalues of all the quantities associated with them, as we will explore in the forthcoming. Throughout we will work only over algebraically closed fields k.

9.1 Polynomials and eigenvalues

From now on, we will assume that $A : V \to V$ is a linear endomorphism of some k-vector space V of dimension $n \geqslant 1$. We recall the following definitions.

Definition 9.1. Recall that an *eigenvector* $v \neq 0$ of a linear map A is a vector such that $Av = \lambda v$ for some $\lambda \in k$. Here λ is called the corresponding *eigenvalue*. The *characteristic polynomial* of A is given by $p_A(x) = \det(xI_n - A)$, which is monic of degree n. By construction,

$$\lambda \text{ is an eigenvalue of } A \iff p_A(\lambda) = 0.$$

The *analytic multiplicity* $a(\lambda)$ of the eigenvalue λ is defined as the multiplicity of the zero of $p_A(x)$ at $x = \lambda$. The *geometric multiplicity* $g(\lambda)$ of the eigenvalue λ is defined as the dimension of the λ-eigenspace of A; in other words

$$g(\lambda) := \dim(E_\lambda) \quad \text{where} \quad E_\lambda = \{v \in V \mid Av = \lambda v\}.$$

The eigenvalues of an upper- or lower-triangular matrix can be read off from its diagonal. The following theorem is very famous.

Theorem 9.2. (Cayley-Hamilton) *It holds that* $p_A(A) = 0$.

The multiplicities $a(\lambda)$ and $g(\lambda)$ are related to A being diagonalisable or not; recall that this means that we can find a basis of V consisting of eigenvectors of A.

Proposition 9.3. *Let $A : V \to V$ be a linear map and let λ be an eigenvalue of A. Then $g(\lambda) \leqslant a(\lambda)$. A matrix is diagonalisable if and only if $g(\lambda) = a(\lambda)$ for all eigenvalues λ.*

The following proposition is sometimes useful.

Proposition 9.4. *Let $A : V \to V$ be a linear map and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A occuring with their algebraic multiplicities. Then for any polynomial $p \in k[X]$, the eigenvalues of the matrix $p(A)$ are given by $p(\lambda_1), \ldots, p(\lambda_n)$.*

It also admits a converse, which is given by the following and is often even more useful than the Cayley-Hamilton Theorem.

Proposition 9.5. *Let $A : V \to V$ be a linear map and let $p \in k[X]$ be a polynomial such that $p(A) = 0$. Then all eigenvalues of A are zeroes of p.*

It is important to note here that not every zero of p must necessarily be an eigenvalue of A and that the multiplicities of the zeroes of p need not necessarily match those of λ as an eigenvalue of A. However, there exists a *minimal polynomial* $\mu_A \in k[X]$ of A with the property that

$$p(A) = 0 \iff \mu_A \mid p.$$

Now it is true that any zero of μ_A must necessarily be an actual eigenvalue of A, but the multiplicities need still not match. Finally, we have the following.

Proposition 9.6. *A matrix A is diagonalisable if and only if the polynomial $\mu_A \in k[X]$ is squarefree; i.e. if it only has simple zeroes in an algebraic closure of k.*

9.2 Trace and determinant

The following two concepts capture a lot of information about a matrix.

Definition 9.7. Let $A = (a_{ij})_{i,j=1}^n$ be an $n \times n$-matrix. Then we define

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii} \quad \text{and} \quad \det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

These quantities enjoy the following familiar properties.

Proposition 9.8. *Let A and B be $n \times n$-matrices. Then the following statements hold.*

- *For any $c \in k$, it holds that $\mathrm{tr}(cA) = c\mathrm{tr}(A)$ and $\det(cA) = c^n \det(A)$.*
- *It holds that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and $\det(AB) = \det(A)\det(B)$.*
- *Swapping two columns or two rows of a matrix multiplies the determinant by -1. Adding one column a number of times to another, or adding one row a number of times to another, leaves the determinant invariant.*

Proposition 9.9. *For any linear endomorphism A, it holds that*

$$\mathrm{tr}(A) = \sum_{\lambda \text{ e.v.}} \lambda \quad \textit{and} \quad \det(A) = \prod_{\lambda \text{ e.v.}} \lambda.$$

In other words, information about the eigenvalues of A can give you valuable information about its trace and its determinant. Recall that a matrix A is called *nilpotent* if $A^k = 0$ for some $k \geqslant 0$. We have many different ways of identifying these matrices.

Proposition 9.10. *For an $n \times n$-matrix A, the following are equivalent:*

- A *is nilpotent;*
- $A^n = 0$*;*
- *All eigenvalues of* A *are equal to* 0*;*
- $\mathrm{tr}(A^k) = 0$ *for all* $1 \leqslant k \leqslant n$.

Finally, it is sometimes useful to not forget that $\mathrm{tr}(A^T) = A$ and $\det(A^T) = \det(A)$. We conclude by recalling that transposition is a covariant operation; in other words, it holds that $(AB)^T = B^T A^T$ for any two square matrices A and B.

9.3 Examples

Example 9.11. *(IMC 2011)* Find all real 3×3-matrices A satisfying the conditions that $\mathrm{tr}(A) = 0$ and in addition $A^2 + A^T = I_3$.

Solution: We are given the equation $A^T = I - A^2$. If we transpose the original equation, we find that

$$(A^T)^2 + A = I \implies (I - A^2)^2 + A = I \implies A^4 - 2A^2 + A = 0.$$

This means that all eigenvalues of A must be roots of the polynomial

$$X^4 - 2X^2 + X = X(X^3 - 2X + 1) = X(X-1)(X^2 + X - 1).$$

In other words, if λ is an eigenvalue of A, then

$$\lambda \in S := \left\{ 0, 1, \frac{-1 \pm \sqrt{5}}{2} \right\}.$$

The same holds for the matrix A^T, and as such, the eigenvalues of $I - A^T$ must all be contained in the set

$$1 - S = \left\{ 1, 0, \frac{1 \pm \sqrt{5}}{2} \right\}.$$

But as $I - A^T = A^2$, these are also the possible eigenvalues of A^2. However, the only eigenvalues of this matrix are the eigenvalues

$$S^2 = \left\{ 0, 1, \frac{3 \pm \sqrt{5}}{2} \right\}.$$

Therefore, the only possible eigenvalues for A can be 0 and 1. However, But then

$$2\mathrm{tr}(A) = \mathrm{tr}(A^2) + \mathrm{tr}(A^T) = \mathrm{tr}(I) = 3;$$

this is a contradiction. Therefore, no such matrices A exist. $\triangle$

Example 9.12. *(IMC 2009)* Let A and B be two square matrices of the same size that satisfy the equation $A^2B + BA^2 = 2ABA$. Show that $AB - BA$ is nilpotent.

Solution: We may rewrite the given equation to

$$A(AB - BA) = (AB - BA)A;$$

in other words, we know that A and $AB - BA$ commute. We then find for any positive integer k that

$$\begin{aligned}
\operatorname{tr}\left((AB-BA)^{k+1}\right) &= \operatorname{tr}\left(AB(AB-BA)^k - BA(AB-BA)^k\right) \\
&= \operatorname{tr}\left(AB(AB-BA)^k\right) - \operatorname{tr}\left(B(AB-BA)^kA\right) \\
&= \operatorname{tr}\left(AB(AB-BA)^k\right) - \operatorname{tr}\left(AB(AB-BA)^k\right) = 0;
\end{aligned}$$

where we used cyclicity of the trace. Since $\operatorname{tr}\left((AB-BA)^k\right) = 0$ for each positive integer k, it follows that $AB - BA$ must be nilpotent, as desired. △

9.4 Exercises

Problem 9.1. *Let P be a square matrix with the property that the sum of the entries in each column equal 1. Show that 1 is an eigenvalue of P.*

Problem 9.2. *Show that there are no square matrices A and B satisfying $AB - BA = I$.*

Problem 9.3. *Let A be a real $n \times n$-matrix with the property that $A^2 + I_n = 0$. What are all possible values of $\det(A)$?*

Problem 9.4. *Let A be a real $n \times n$-matrix with n odd. Can $A - A^T$ be invertible?*

Problem 9.5. *A real 3×3-matrix A satisfies*

$$A^2 = \begin{pmatrix} 4 & 4 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Determine all possible values of $\operatorname{tr}(A)$.

Problem 9.6. *Let A be a square matrix with the property that $\operatorname{tr}(X) = 0 \implies \operatorname{tr}(AX) = 0$. Show that A is a diagonal matrix.*

Problem 9.7. *Let $a, b, p_1, \dots, p_n$ be real numbers with $a \neq b$. Let*

$$f(x) = (p_1 - x) \cdots (p_n - x).$$

Prove that

$$\det \begin{pmatrix} p_1 & a & \cdots & a & a \\ b & p_2 & \cdots & a & a \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & \cdots & p_{n-1} & a \\ b & b & \cdots & b & p_n \end{pmatrix} = \frac{bf(a) - af(b)}{b - a}.$$

Problem 9.8. *Let* A *be a real* $n \times n$*-matrix with the property that* $a_{ii} = 1$ *and* $a_{ij} + a_{ji} = 1$ *for all* $1 \leqslant i, j \leqslant n$. *Show that* $\det(A) > 0$.

Problem 9.9. $(*)$ *To each number of* n^2 *digits we let* $f(m)$ *denote the determinant of the matrix obtained by writing the digits of* m *in order along the rows. For example,*

$$f(8617) = \det \begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix} = 50.$$

Determine the sum of the values of $f(m)$ *as* m *ranges over all positive integers with* n^2 *digits.*

Problem 9.10. $(*)$ *Let* A *be the* $n \times n$*-matrix given by* $a_{i,i+1} = a_{i+1,i} = i + 1$ *and* $a_{ii} = i^2 + 1$ *for all* $1 \leqslant i < n$*, and finally* $a_{nn} = n^2$*, with zeroes elsewhere. Determine* $\det(A)$.

Problem 9.11. $(*)$ *Let* A, B *be real* $n \times n$*-matrices. Let* A_k *denote the matrix* A *in which the first column of* A *was replaced by the* k*-th column of* B*, and similarly define* B_k. *Show that*

$$\det(AB) = \sum_{k=1}^{n} \det(A_k B_k).$$

9.5 IMC Problems

Problem 9.12. (1994) *Let* $\alpha \in \mathbb{R}^\times$ *and let* A *and* B *be real* $n \times n$*-matrices satisfying the property that* $AB - BA = \alpha A$. *Show that for all* $k \geqslant 1$*, we have that* $A^k B - BA^k = \alpha k A^k$ *and use this to prove that* A *is nilpotent.*

Problem 9.13. (1997) *Let* M *be an invertible* $2n \times 2n$*-matrix, represented in block form as*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \textit{and} \quad M^{-1} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}.$$

Show that $\det(M) \cdot \det(H) = \det(A)$.

Problem 9.14. (1997) *Let* $f : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathbb{R}$ *be a linear map. Show that there exists a unique matrix* C *such that* $f(A) = \mathrm{tr}(AC)$ *for all* $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$. *Show that if further* $f(AB) = f(BA)$ *for all* $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{R})$*, then* $f(A) = \lambda \cdot \mathrm{tr}(A)$ *for all* $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$.

Problem 9.15. (1999) *Show that for any positive integer* n*, there exists a real* $n \times n$*-matrix* A *satisfying* $A^3 = A + I_n$. *Show that any such matrix must satisfy that* $\det(A) > 0$.

Problem 9.16. (2000) *Let* A *and* B *be square matrices of the same size and suppose that* $\mathrm{rk}(AB - BA) = 1$. *Show that* $(AB - BA)^2 = 0$.

Problem 9.17. (2002) *Let* A *be the* $n \times n$*-matrix given by* $a_{ij} = (-1)^{i+j}$ *if* $i \neq j$*, and* $a_{ii} = 2$ *for all* $1 \leqslant i, j \leqslant n$. *Determine* $\det(A)$.

Problem 9.18. (2003) *Let* A *be an* $n \times n$*-matrix satisfying the equation* $3A^3 = A^2 + A + I_n$. *Show that the sequence* A^k *converges to an idempotent matrix.*

Problem 9.19. (2005) *Given* $n \geqslant 1$, *find the largest possible dimension of a subspace* V *of the space of all real* $n \times n$*-matrices with the property that* $\operatorname{tr}(AB) = 0$ *for all* $A, B \in V$.

Problem 9.20. (2007) *Do there exist real* 2×2*-matrices* A, B, C *such that*

$$\det(xA + yB + zC) = x^2 + y^2 + z^2 \quad \text{for all } x, y, z \in \mathbb{R}?$$

Problem 9.21. (2007) *Let* $n > 1$ *be an odd positive integer and let* A *be the matrix defined by* $a_{ii} = 2$ *and* $a_{ij} = 2$ *if* $i - j \equiv \pm 2 \mod n$, *with zeroes elsewhere. Determine* $\det(A)$.

Problem 9.22. (2008)($*$) *Let* n *be a positive integer and consider the matrix* A *given by* $a_{ij} = 1$ *if* $i + j$ *is prime and* $a_{ij} = 0$ *otherwise. Prove that* $|\det(A)|$ *is a perfect square.*

Problem 9.23. (2013) *Let* A *and* B *be real symmetric matrices all of whose eigenvalues are strictly greater than 1. Let* λ *be an eigenvalue of* AB. *Show that* $|\lambda| > 1$.

Problem 9.24. (2014) *Determine all pairs* (a, b) *of real numbers for which there exists a unique real symmetric* 2×2*-matrix* M *satisfying* $\operatorname{tr}(M) = a$ *and* $\det(M) = b$.

Problem 9.25. (2014) *Let* A *be a real symmetric matrix with eigenvalues* $\lambda_1, \ldots, \lambda_n$.
Show that

$$\sum_{1 \leqslant i < j \leqslant n} a_{ii} a_{jj} \geqslant \sum_{1 \leqslant i < j \leqslant n} \lambda_i \lambda_j.$$

When does equality hold?

Problem 9.26. (2015) *Let* $n \geqslant 2$ *and let* A *and* B *be real* $n \times n$*-matrices satisfying the equation* $A^{-1} + B^{-1} = (A + B)^{-1}$. *Prove that* $\det(A) = \det(B)$. *Does the same conclusion follow if* A *and* B *are allowed to be complex matrices?*

Problem 9.27. (2017) *Which complex numbers can occur as the eigenvalue of a real square matrix* A *satisfying* $A^2 = A^T$?

Problem 9.28. (2020) *Let* A *and* B *be real* $n \times n$*-matrices such that* $\operatorname{rk}(AB - BA + I) = 1$. *Prove that* $\operatorname{tr}(ABAB) - \operatorname{tr}(A^2B^2) = n(n-1)/2$.

Problem 9.29. (2021) *Let* A *be a real square matrix with the property that for every* $m \geqslant 1$, *there exists a real symmetric matrix* B *such that* $2021B = A^m + B^2$. *Prove that* $|\det(A)| \leqslant 1$.

Problem 9.30. (2022) *Let* n *be a positive integer. Find all real* $n \times n$ *matrices* A *with only real eigenvalues satisfying* $A + A^k = A^T$ *for some integer* $k \geqslant n$.

Problem 9.31. (2024) *Ivan writes the matrix* $\begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$ *on the board. Then he performs the following operation on the matrix several times:*

- *he chooses a row or a column of the matrix, and*
- *he multiplies or divides the chosen row of column entry-wise by the other row or column, respectively.*

Can Ivan end up with the matrix $\begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$ *after finitely many steps?*

10 Analysis IV

We conclude our discussions about analysis with some more advanced concepts involving functions, including clever substitutions, functional equations and inequalities.

10.1 Goniometric functions

Goniometric functions, those derived from sin and cos, are more arithmetic than one would guess at first sight. In the previous section we already saw their definitions in terms of power series expansions. Using complex numbers, they are much more easily defined;

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}.$$

Sometimes, their real analogs pop up, the hyperbolic functions, which are given by

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \qquad \cosh(x) = \frac{e^{x} + e^{-x}}{2}.$$

Most effective are applications of goniometric functions when recognising their particular formulas, of which we list a few here:

$$\begin{aligned}
\sin(x)^2 + \cos(x)^2 &= 1; \\
\sin(x \pm y) &= \sin(x)\cos(y) \pm \cos(x)\sin(y); \\
\cos(x \pm y) &= \cos(x)\cos(y) \mp \sin(x)\sin(y); \\
\tan(x \pm y) &= \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}; \\
\sin(2x) &= 2\sin(x)\cos(x); \\
\cos(2x) &= 2\cos(x)^2 - 1; \\
\tan(2x) &= \frac{2\tan(x)}{1 - \tan(x)^2}; \\
\sin(x) + \sin(y) &= 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}; \\
\sin(x) - \sin(y) &= 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}; \\
\cos(x) + \cos(y) &= 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}; \\
\cos(x) - \cos(y) &= -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}.
\end{aligned}$$

It can be difficult to spot when a clever goniometric substitution is the right way to move forward, but for example, expressions like $2x^2 - 1$ can hint at the involvement of a cosine. Similarly, the sudden appearance of a π usually hints at some underlying goniometric structure waiting to be uncovered and exploited.

10.2 Functional equations

Functional equations are quite a classical topic, even for high school olympiads, asking to determine all functions satisfying a certain set of conditions. The following functional equation in particular is very famous.

Proposition 10.1. (Cauchy's Equation) *Let* $f : \mathbb{Q} \to \mathbb{R}$ *be a function satisfying the equation* $f(x+y) = f(x) + f(y)$ *for all* $x, y \in \mathbb{Q}$*. Then* $f(x) = cx$ *for some* $c \in \mathbb{R}$*.*

It is to be stressed that using the axiom of choice, one can construct very nasty and discontinuous solutions to this equation for general functions $f : \mathbb{R} \to \mathbb{R}$, but on $\mathbb{Q}$, everything works nicely. Combined with a continuity condition, the same condition follows on all of $\mathbb{R}$ too.

A common technique for solving functional equations is repeatedly substituting expressions into themselves. If you can spot an expression $g(x)$ in your functional equation involving the mystery function f that has the property that $g(g(\ldots(g(x))\ldots)) = x$, one often deduces interesting information. Common examples are

$$g(x) = 1 - x \quad \text{with} \quad g(g(x)) = x, \quad \text{and} \quad g(x) = \frac{1}{1-x}, \quad \text{with} \quad g(g(g(x))) = x.$$

At the IMC, contrary to on the high school olympiads, functional equations are often paired with derivatives or integrals to create some interesting differential equations. It is important to remember the solutions to some of the most basic ones you can encounter, like

$$f(x) = \alpha \cdot f'(x) \implies f(x) = c \cdot e^{\alpha x}.$$

It is also imperative to remember that

$$\frac{d}{dx} \log(f(x)) = \frac{f'(x)}{f(x)}.$$

Further tips for solving functional equations is trying to prove that f is surjective or injective, as this sometimes allows you to cancel f's from your expressions. If you encounter the expression $f(f(x))$, it might give you additional information to consider $f(f(f(x)))$ in two different ways. It can also sometimes be useful to determine the set of x for which $f(x) = x$. Never forget to apply induction in case you have some additive condition on the argument of your function; sometimes this can be used in combination with continuity in powerful ways. If your functional equation displays some symmetry, it is often a good idea to exploit that. Finally, the single most important thing to do when faced with a functional equation is to try plugging in some explicit values of the variables in your expressions; substituting $x = 0$ or $x = y$ can often reveal some powerful equations hiding in the mess of the full condition.

10.3 Inequalities

One of the easiest inequalities about functions comes from analysing its derivative.

Proposition 10.2. *Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function that is continuously differentiable on* (a, b)*. Suppose that* $f'(x) \geqslant 0$ *for all* $x \in (a, b)$*. Then* $f(b) \geqslant f(a)$ *with equality if and only if* $f' = 0$*.*

However, there are also more profound inequalities about functions, of which the following is quite well known.

Theorem 10.3. (Jensen's Inequality) *Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function that is twice differentiable on* (a, b)*. Suppose that* $f''(x) \geqslant 0$ *for all* $x \in (a, b)$*. Let* $x_1, \ldots x_n \in [a, b]$ *be arbitrary and let* $\lambda_1, \ldots, \lambda_n \geqslant 0$ *satisfy* $\lambda_1 + \ldots + \lambda_n = 1$*. Then*

$$f(\lambda_1 x_1 + \ldots + \lambda_n x_n) \leqslant \lambda_1 f(x_1) + \ldots + \lambda_n f(x_n).$$

In particular, it holds that

$$f\left(\frac{x_1 + \ldots + x_n}{n}\right) \leqslant \frac{f(x_1) + \ldots + f(x_n)}{n}.$$

This implies the following continuous version:

$$\text{if } \int_C \lambda(x)\,dx = 1, \text{ then } \quad f\left(\int_C x\lambda(x)\,dx\right) \leqslant \int_C f(x)\lambda(x)\,dx.$$

There is a continuous analogon of the generalised inequality about arithmetic means.

Theorem 10.4. *Let* $p \geqslant q$ *and let* f *be integrable on a measureable set* $C \subset \mathbb{R}$*. Then*

$$\left(\int_C |f(x)|^p\,dx\right)^{1/p} \geqslant \left(\int_C |f(x)|^q\,dx\right)^{1/q},$$

with equality only when $p = q$ *or* f *is constant.*

The following is another restatement of the Cauchy-Schwarz inequality.

Corollary 10.5. *Let* f, g *be integrable functions on a measureable set* $C \subset \mathbb{R}$*. Then*

$$\left(\int_C f(x)g(x)\,dx\right)^2 \leqslant \left(\int_C |f(x)|^2 dx\right)\left(\int_C |g(x)|^2 dx\right).$$

Finally, Hölder's inequality combines the above two.

Theorem 10.6. (Hölder's inequality) *Let* $p, q \geqslant 1$ *satisfy* $1/p + 1/q = 1$*. Then for any two integrable functions on some measureable subset* $C \subset \mathbb{R}$*, it holds that*

$$\int_C |f(x)g(x)|dx \leqslant \left(\int_C |f(x)|^p\,dx\right)^{1/p}\left(\int_C |g(x)|^q\,dx\right)^{1/q}.$$

Equality holds only when $|f|^p$ *and* $|g|^q$ *agree up to a fixed scalar.*

10.4 Examples

Example 10.7. *(IMC 2022)* Let $f : [0,1] \to (0,\infty)$ be an integrable function with the property that $f(x) \cdot f(1-x) = 1$ for all $x \in [0,1]$. Prove that

$$\int_0^1 f(x)dx \geqslant 1.$$

Solution: We deduce the inequalities

$$\left(\sqrt{f(x)} - \sqrt{f(1-x)}\right)^2 \geqslant 0 \implies f(x) + f(1-x) \geqslant 2\sqrt{f(x)f(1-x)} = 2.$$

By symmetry, we have that

$$\int_0^1 f(x)\, dx = \int_0^1 f(1-x)\, dx.$$

As such, we find that

$$\int_0^1 f(x)\, dx = \int_0^1 \frac{f(x)+f(1-x)}{2}\, dx \geqslant \int_0^1 1\, dx = 1,$$

completing the proof. △

Example 10.8. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with

$$f(0) = 1 \quad \text{and} \quad f(2x) - f(x) = x$$

for all $x \in \mathbb{R}$.

Solution: We rewrite the given equation to $f(x/2) = f(x) - x/2$ for all $x \in \mathbb{R}$. We may then iterate this to find that

$$\begin{aligned}
f(x/4) &= f(x/2) - x/4 = f(x) - x/2 - x/4;\\
f(x/8) &= f(x) - x/2 - x/4 - x/8;\\
&\cdots\\
f(x/2^n) &= f(x) - \sum_{k=1}^{n} \frac{x}{2^k} = f(x) - \frac{2^n-1}{2^n}x.
\end{aligned}$$

If we now let $n \to \infty$, using the supposed continuity of f, we obtain that

$$1 = f(0) = f(x) - x.$$

As such, $f(x) = x + 1$ for all $x \in \mathbb{R}$. △

10.5 Exercises

Problem 10.1. *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *with*

$$f(x) + xf(1 - x) = 1 + x \quad \textit{for all } x \in \mathbb{R}.$$

Problem 10.2. *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$f(x) = 1 + \int_0^x f(t)dt.$$

Problem 10.3. *Determine all functions* $f : \mathbb{Z} \to \mathbb{Z}$ *satisfying* $f(1) = 3$ *and*

$$f(x + y) = f(x) + f(y) + 1 \quad \textit{for all } x, y \in \mathbb{Z}.$$

Problem 10.4. *Determine all increasing and strongly multiplicative (i.e.* $f(mn) = f(m)f(n)$ *for all* m *and* n*) functions* $f : \mathbb{N} \to \mathbb{N}$ *satisfying* $f(2) = 2$.

Problem 10.5. *A continuous function* $f : \mathbb{R} \to \mathbb{R}$ *satisfies that* $f(x) = f(x^2)$ *for all* $x \in \mathbb{R}$. *Prove that* f *is constant.*

Problem 10.6. *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *that are continuous at* 0*, satisfying*

$$f(x) = f\left(\frac{x}{1 - x}\right) \quad \textit{for all } x \neq 1.$$

Problem 10.7. *Find all functions* $f : \mathbb{Z} \to \mathbb{Z}$ *with* $f(0) = 1$ *satisfying*

$$f(f(n)) = n \quad \textit{and} \quad f(f(n + 2) + 2) = n \quad \textit{for all } n \in \mathbb{Z}.$$

Problem 10.8. *Determine all continuous functions* $f : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$ *such that for all* $x \in \mathbb{R} \setminus \{0, 1\}$*, it holds that*

$$f(x) + f\left(\frac{x - 1}{x}\right) = x + 1.$$

Problem 10.9. *Determine all functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$f(x)f(y) = f(x) + yf(x) \quad \textit{for all } x, y \in \mathbb{R}.$$

Problem 10.10. *Consider a continuous function* $f : \mathbb{R} \to \mathbb{R}$ *satisfying the property that*

$$f(2x^2 - 1) = 2xf(x) \quad \textit{for all } x \in [0, 1].$$

Show that $f = 0$ *on* $[-1, 1]$.

Problem 10.11. *Determine all continuously differentiable functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying for all* $x \in \mathbb{R}$ *the equation*

$$f(x)^2 = \int_0^x f(t)^2 + f'(t)^2 dt + 2023.$$

Problem 10.12. *Determine all functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying for all* $x, y \in \mathbb{R}$ *the equation*

$$f(x^2 - y^2) = (x - y)(f(x) + f(y)).$$

Problem 10.13. *Consider a continuous function* $f : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$f(2x - f(x)) = x \quad \textit{for all } x \in \mathbb{R}.$$

Suppose that $f(\alpha) = \alpha$ *for some* $\alpha \in \mathbb{R}$. *Show that then* $f(x) = x$ *for all* $x \in \mathbb{R}$. *Does this conclusion still hold if* f *such an* α *need not necessarily exist?*

Problem 10.14. *Let* $c > 0$. *Determine all continuous functions* $f : \mathbb{R} \to \mathbb{R}$ *with the property that*

$$f(x) = f(x^2 + c) \quad \textit{for all } x \in \mathbb{R}.$$

Problem 10.15. *Determine all* $\alpha > 0$ *with the property that for all differentiable functions* $f : \mathbb{R} \to \mathbb{R}_{>0}$ *such that* $f'(x) > f(x)$ *for all* $x \in \mathbb{R}$, *there exists some* $N \in \mathbb{R}$ *such that* $f(x) > e^{\alpha x}$ *for all* $x > N$.

Problem 10.16. *Let* $a, b \in (0, 1/2)$ *and let* g *be a continuous function satisfing the property that*

$$g(g(x)) = ag(x) + bx \quad \textit{for all } x \in \mathbb{R}.$$

Prove that $g(x) = cx$ *for some* $c \in \mathbb{R}$.

Problem 10.17. *Prove that there exists no function* $f : \mathbb{R} \to \mathbb{R}$ *satisying*

$$f(f(x) + y) = f(x) + 3x + yf(y) \quad \textit{for all } x, y \in \mathbb{R}.$$

Problem 10.18. *Determine all differentiable functions* $f : (0, \infty) \to (0, \infty)$ *for which there exists some* $a \in \mathbb{R}$ *such that*

$$f'(a/x) = x/f(x) \quad \textit{for all } x > 0.$$

Problem 10.19. *Determine all functions* $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ *satisfying*

$$xf(x^2 f(y)) = yf(x) \quad \textit{for all } x, y \in \mathbb{R}.$$

Problem 10.20. *Determine all twice differentiable functions* $f : \mathbb{R} \to \mathbb{R}$ *with* $f'(x)f''(x) = 0$ *for all* $x \in \mathbb{R}$.

Problem 10.21. *Determine all continuous functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$f(x) + \int_0^x (x - t)f(t)dt = 1 \quad \textit{for all } x \in \mathbb{R}.$$

Problem 10.22. *Let* $f : [-2, 2] \to \mathbb{R}$ *be continuously differentiable. Prove that there exists some* $x \in (-2, 2)$ *such that*

$$f'(x) - f(x)^2 < 1.$$

Problem 10.23. $(*)$ *Determine all functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$|x|f(y) + yf(x) = f(xy) + f(x^2) + f(f(y)) \quad \textit{for all } x, y \in \mathbb{R}.$$

Problem 10.24. $(*)$ *Let* $a > 0$. *For which value of* a *is the integral*

$$\int_a^{a^2} \frac{1}{x + \sqrt{x}} dx \quad \textit{minimal?}$$

Problem 10.25. $(*)$ *Consider the set of continuously differentiable functions* $f : [0, 1] \to \mathbb{R}$ *satisfying* $f(0) = 0$ *and* $f(1) = 1$. *Determine the minimal value of*

$$\int_0^1 |f'(x) - f(x)| dx.$$

Problem 10.26. $(*)$ *Let* $f : [0, 1] \to \mathbb{R}$ *be continuous. Prove that*

$$\int_0^1 \int_0^1 |f(x) + f(y)| dx dy \geqslant \int_0^1 |f(x)| dx.$$

Problem 10.27. $(*)$ *Let* $f(x, y) : [0, 1] \times [0, 1] \to \mathbb{R}$ *be continuous. Define*

$$A = \int_0^1 \left(\int_0^1 f(x, y) dx \right)^2 dy + \int_0^1 \left(\int_0^1 f(x, y) dy \right)^2 dx$$

and similarly the quantity

$$B = \left(\int_0^1 \int_0^1 f(x, y) dx dy \right)^2 + \int_0^1 \int_0^1 f(x, y)^2 dx dy.$$

Prove that $A \leqslant B$.

10.6 IMC Problems

Problem 10.28. (1994) *Let* $f : (a, b) \to \mathbb{R}$ *be a continuously differentiable function with the properties that*

$$\lim_{x \to a} f(x) = \infty, \quad \lim_{x \to b} f(x) = -\infty \quad \textit{and} \quad f'(x) + f(x)^2 \geqslant -1 \quad \textit{for all } x \in (a, b).$$

Show that $b - a \geqslant \pi$ *and find all possible* f *for which equality holds.*

Problem 10.29. (1994) *Let* $f : [a, b] \to \mathbb{R}$ *be continuously differentiable with* $f(a) = 0$. *Suppose that for some* $\lambda > 0$, *it holds that*

$$|f'(x)| \leqslant \lambda |f(x)| \quad \textit{for all } x \in [a, b].$$

Prove that $f = 0$.

Problem 10.30. (1995) *Let* $f : [0,1] \to \mathbb{R}$ *be a continuous function with that for every* $x \in [0,1]$, *it holds that*

$$\int_x^1 f(t)dt \geqslant \frac{1-x^2}{2}.$$

Show that

$$\int_0^1 f(t)^2 dt \geqslant \frac{1}{3}.$$

Problem 10.31. (1996) *Let* $n \in \mathbb{N}$. *Evaluate the integral*

$$\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin(x)}.$$

Problem 10.32. (1998) *Let* $f : [0,1] \to \mathbb{R}$ *be a continuous function with the property that for any* $x, y \in [0,1]$, *it holds that*

$$xf(y) + yf(x) \leqslant 1.$$

Show that

$$\int_0^1 f(x)dx \leqslant \frac{\pi}{4}.$$

Can equality hold?

Problem 10.33. (1999) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function satisfying*

$$\Big|\sum_{k=1}^{n} 3^k \big(f(x+ky) - f(x-ky)\big)\Big| \leqslant 1,$$

for all $n \in \mathbb{N}$ *and all* $x, y \in \mathbb{R}$. *Prove that* f *must be a constant function.*

Problem 10.34. (1999) *Find all strictly monotonous functions* $f : (0,\infty) \to (0,\infty)$ *satisfying*

$$f(x^2/f(x)) = x \quad \textit{for all } x > 0.$$

Problem 10.35. (1999) *Determine all functions* $f : (0,\infty) \to (0,\infty)$ *satisfying*

$$f(x)^2 \geqslant f(x+y)(f(x)+y) \quad \textit{for all } x, y > 0.$$

Problem 10.36. (2000) *Find all functions* $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ *satisfying for all* $x, y > 0$ *the equation*

$$f(x)f(yf(x)) = f(x+y).$$

Problem 10.37. (2001) *Prove that there is no function* $f : \mathbb{R} \to \mathbb{R}$ *satisfying both* $f(0) > 0$ *and*

$$f(x+y) \geqslant f(x) + yf(f(x)) \quad \textit{for all } x, y \in \mathbb{R}.$$

Problem 10.38. (2001)($*$) *For each positive integer* n, *let* $f_n(\theta) = \sin(\theta)\sin(2\theta)\sin(4\theta)\cdots\sin(2^n\theta)$. *Show that*

$$|f_n(\theta)| \leqslant \frac{2}{\sqrt{3}}|f_n(\pi/3)| \quad \textit{for all } \theta \in \mathbb{R} \textit{ and } n \in \mathbb{N}.$$

Problem 10.39. (2002) *Does there exist a continuously differentiable function* $f : \mathbb{R} \to \mathbb{R}$ *such that for every* $x \in \mathbb{R}$ *we have both*

$$f(x) > 0 \quad \textit{and} \quad f'(x) = f(f(x))?$$

Problem 10.40. (2003) *For any choice of* $m, n \geqslant 1$, *evaluate the limit*

$$\lim_{x \to 0} \int_x^{2x} \frac{\sin(t)^m}{t^n} dt.$$

Problem 10.41. (2004) *Prove that*

$$\int_0^1 \int_0^1 \frac{dx dy}{1/x + |\log(y)| - 1} \leqslant 1.$$

Problem 10.42. (2005)($*$) *Let* $f : \mathbb{R} \to [0, \infty)$ *be a continuously differentiable function. Prove that*

$$\left| \int_0^1 f(x)^3 dx - f(0)^2 \int_0^1 f(x) dx \right| \leqslant \max_{0 \leqslant x \leqslant 1} |f'(x)| \left(\int_0^1 f(x) dx \right)^2 .$$

Problem 10.43. (2005) *Let* $f : (0, \infty) \to \mathbb{R}$ *be a twice continuously differentiable function such that*

$$|f''(x) + 2xf'(x) + (x^2 + 1)f(x)| \leqslant 1$$

for all $x > 0$. *Prove that* $\lim_{x \to \infty} f(x) = 0$.

Problem 10.44. (2006) *For any given* $x \in (0, \pi/2)$ *determine which of* $\tan(\sin(x))$ *and* $\sin(\tan(x))$ *is bigger.*

Problem 10.45. (2009) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a twice differentiable function satisfying* $f(0) = 1$ *and* $f'(0) = 0$, *and further*

$$f''(x) - 5f'(x) + 6f(x) \geqslant 0 \quad \textit{for all } x \geqslant 0.$$

Show that for all $x \geqslant 0$, *it holds that*

$$f(x) \geqslant 3e^{2x} - 2e^{3x}.$$

Problem 10.46. (2010) *Let* $0 < a < b$. *Prove that*

$$\int_a^b (x^2 + 1)e^{-x^2} dx \geqslant e^{-a^2} - e^{-b^2}.$$

Problem 10.47. (2012) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuously differentiable function with the property that*

$$f'(t) > f(f(t)) \quad \textit{for all } t \in \mathbb{R}.$$

Show that

$$f(f(f(t))) \leqslant 0 \quad \textit{for all } t \geqslant 0.$$

Problem 10.48. (2013) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a twice differentiable function with the property that* $f(0) = 0$. *Show that there exists some* $c \in (-\pi/2, \pi/2)$ *such that*

$$f''(c) = f(c)(1 + 2\tan(c)^2).$$

Problem 10.49. (2014)($*$) *Define* $f(x) = \sin(x)/x$. *Show that for all* $x > 0$ *and* $n \geqslant 1$, *it holds that*

$$|f^{(n)}(x)| < \frac{1}{n+1},$$

where $f^{(n)}$ *denotes the* n*th derivative of* f.

Problem 10.50. (2016) *Consider the set of continuous functions* $f : [0, 1] \to \mathbb{R}$ *satisfying*

$$f(x) + f(y) \geqslant |x - y| \quad \textit{for all } x, y \in [0, 1].$$

Find the minimal value of

$$\int_0^1 f(x)\,dx$$

for functions in this set.

Problem 10.51. (2017) *Consider a differentiable function* $f : \mathbb{R} \to (0, \infty)$ *with the property that for some constant* $L > 0$, *it holds that*

$$|f'(x) - f'(y)| \leqslant L|x - y| \quad \textit{for all } x, y \in \mathbb{R}.$$

Show that for all $x \in \mathbb{R}$, *it holds that*

$$f'(x)^2 < 2Lf(x).$$

Problem 10.52. (2018) *Find all differentiable functions* $f : (0, \infty) \to \mathbb{R}$ *with the property that*

$$f(b) - f(a) = (b - a)f'(\sqrt{ab}) \quad \textit{for all } a, b > 0.$$

Problem 10.53. (2019) *Let* $f : (-1, 1) \to \mathbb{R}$ *be a twice differentiable function such that* $2f'(x) + xf''(x) \geqslant 1$ *for all* $x \in (-1, 1)$. *Prove that*

$$\int_{-1}^1 xf(x)\,dx \geqslant \frac{1}{3}.$$

Problem 10.54. (2020) *Find all twice continuously differentiable functions* $f : \mathbb{R} \to (0, \infty)$ *such that*

$$f''(x)f(x) \geqslant 2f'(x)^2 \quad \textit{for all } x \in \mathbb{R}.$$

Problem 10.55. (2023) *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *that have a continuous second derivative and for which the equality*

$$f(7x + 1) = 49f(x) \quad \textit{holds for all } x \in \mathbb{R}.$$

11 Algebra IV

We conclude our treatment of algebra by discussing some somewhat more advanced ideas surrounding the theory of polynomials that can sometimes be useful to settle subtle questions about these rich objects.

11.1 Advanced ideas on polynomials

We know that $n+1$ distinct values uniquely determine a degree n polynomial. Explicitly, given $x_0, \dots x_n \in \mathbb{C}$ and $y_0, \dots, y_n \in \mathbb{C}$, the Lagrange-polynomial

$$p(x) = \sum_{i=0}^{n} y_i \prod_{\substack{0 \leqslant j \leqslant n \\ j \neq i}} \frac{x - x_j}{x_i - x_j}$$

satisfies $p(x_i) = y_i$ for all $0 \leqslant i \leqslant n$. More analytically, we can take derivatives of polynomials. If $p(x) = c(x - r_1) \cdots (x - r_n)$ as above, then by the product rule,

$$p'(x) = p(x) \sum_{i=1}^{n} \frac{1}{x - r_i}.$$

From this, it follows that if α is a zero of p with multiplicity $k \geqslant 1$, then α is a zero of p' with multiplicity $k-1$. We have the following theorem.

Theorem 11.1. (Gauss-Lucas Theorem) *For $p \in \mathbb{C}[x]$, the zeroes of p' are contained in the complex hull of the zeroes of p.*

The space of polynomials of bounded degree is a vector space; it is possible to apply some knowledge from linear algebra to these spaces!

Finally, we remark that polynomials with integer coefficients, those $p(x) \in \mathbb{Z}[x]$, allow for very interesting number theoretical applications. The most basic observation is that

$$a \equiv b \mod n \implies p(a) \equiv p(b) \mod n.$$

Phrased differently, this says that for any $a, b \in \mathbb{Z}$, it holds that

$$a - b \mid p(a) - p(b).$$

For example, if both $p(0)$ and $p(1)$ are odd, then p cannot have any integral zeroes, simply because all its values must be odd. If $g(x) \in \mathbb{Z}[x]$ is monic and we divide p by g with remainder, than both $q(x)$ and $r(x)$ from Lemma **??** have integral coefficients as well. In particular, the polynomial

$$\frac{p(x) - p(a)}{x - a}$$

is also a member of $\mathbb{Z}[x]$, which is sometimes useful to know. Finally, we remark that if it is known that $p(a_i) = 0$ for some $\{a_1, \ldots, a_k\}$, then

$$(x - a_1) \ldots (x - a_k) \mid p(x);$$

this can have striking applications to the possible values $p(x)$ can take for a given x.

11.2 Examples

Example 11.2. Does there exist some $p(x) \in \mathbb{Z}[x]$ such that $p(0)p(2) = -2$?

Solution: By the above, it follows that $2 \mid p(2) - p(0)$, so $p(0)$ and $p(2)$ must have the same parity. If they are both odd, so is their product; so they must both be even. However, then their product is a multiple of 4, and thus in particular not -2. Such a polynomial can therefore not exist. $\triangle$

Example 11.3. Does there exist some polynomial $p(x) \in \mathbb{Z}[x]$ such that

$$p(0) = 2, \quad p(2) = 4 \quad \text{and} \quad p(4) = 18?$$

Solution: Suppose that $p \in \mathbb{Z}[x]$ is such that $p(0) = 2$ and $p(2) = 4$. Let $q(x) = p(x) - x - 2 \in \mathbb{Z}[x]$. Then by construction, $q(0) = q(2) = 0$ and as such,

$$x(x - 2) \mid q(x) = p(x) - x - 2.$$

Plugging in $x = 4$ now yields that $8 \mid p(4) - 6$, so in particular $p(4) \neq 18$. $\triangle$

11.3 Exercises

Problem 11.1. *Let $p(x) \in \mathbb{Z}[x]$ be a polynomial such that the equation $p(x) = 1$ has three distinct integer solutions. Prove that p does not have any integer zeroes.*

Problem 11.2. *Let $p(x) \in \mathbb{Z}[x]$ be a polynomial with $p(1) = 44$, $p(5) = 0$ and $p(9) = -12$. Prove that $p(12)$ is divisible by 231.*

Problem 11.3. *Let $p(x) \in \mathbb{Z}[x]$ be a polynomial with the property that there exist four distinct $a, b, c, d \in \mathbb{Z}$ such that $p(a) = p(b) = p(c) = p(d) = 5$. Prove that there exists no integer such that $p(k) = 8$.*

Problem 11.4. *Let $p(x) \in \mathbb{Z}[x]$ be a polynomial with $p(1) = 4$, $p(5) = 16$ and $p(9) = 60$. Prove that p does not have an integral zero.*

Problem 11.5. *Let* $p(x)$ *be the unique degree* n *polynomial satisfying* $p(k) = 2^k$ *for all integers* $0 \leqslant k \leqslant n$. *Determine* $p(n+1)$.

Problem 11.6. *Let* $p(x) \in \mathbb{R}[x]$ *be the unique degree* n *polynomial satisfying* $p(k) = k/(k+1)$ *for all integers* $0 \leqslant k \leqslant n$. *Determine* $p(n+1)$.

Problem 11.7. *Let* r, s *and* t *be the three zeroes of some cubic polynomial* p. *Now let* q *be the unique cubic polynomial satisfying* $q(r) = s + t$, $q(s) = t + r$ *and* $q(t) = r + s$. *Express* $q(0)$ *in the coefficients of* p.

Problem 11.8. *Let* $a, b, c \in \mathbb{R}$ *and let* p *be the unique quadratic polynomial with* $p(a) = bc$, $p(b) = ca$ *and* $p(c) = ab$. *Determine* $p(a + b + c)$.

Problem 11.9. *Let* $p(x) \in \mathbb{Z}[x]$ *be such that* $p(x)$ *and* $p(p(p(x)))$ *share a common zero. Show that they even share a common integral zero.*

Problem 11.10. *Let* $a_1, \ldots, a_n$ *be distinct integers for some* $n \geqslant 2$. *Prove that the polynomial* $p(x) = (x - a_1) \cdots (x - a_n) - 1$ *is irreducible in* $\mathbb{Z}[x]$.

Problem 11.11. *Let* $p(x) \in \mathbb{Z}[x]$ *be a degree 5 polynomial satisfying* $p(0) = 100$, $p(1) = 105$, $p(2) = 110$, $p(3) = 115$ *and* $p(4) > 0$. *Determine the smallest possible value of* $p(4)$.

Problem 11.12. *Let* $n \geqslant 1$ *be an integer. Determine all polynomials* $p(x) \in \mathbb{Z}[x]$ *of degree* n *such that* $p(p(x)) = p(x^n) + p(x) - 1$.

Problem 11.13. *Let* $p(x) \in \mathbb{Z}[x]$ *be a monic polynomial such that* $|p(0)|$ *is not a square. Prove that the polynomial* $p(x^2)$ *is irreducible in* $\mathbb{Z}[x]$.

Problem 11.14. *Let* $p(x) \in \mathbb{Z}[x]$ *be a polynomial with the property that for two distinct* $c, d \in \mathbb{Z}$ *it holds that* $p(c) = d$ *and* $p(d) = c$. *Prove that the equation* $p(x) = x$ *has at most one integer solution.*

Problem 11.15. *Consider* $a_1, a_2, a_3 \in \mathbb{Z}$ *and set* $a_4 = a_1$. *Prove that there exists no polynomial* $p(x) \in \mathbb{Z}[x]$ *with the property that* $p(a_i) = a_{i+1}$ *for* $i \in \{1, 2, 3\}$.

Problem 11.16. *Determine all natural numbers* n *with the property that there exists some polynomial* $p \in \mathbb{Z}[x]$ *such that for all positive divisors* $d \mid n$, *it holds that* $p(d) = (n/d)^2$.

Problem 11.17. *Let* $p(x) \in \mathbb{Z}[x]$ *be a monic polynomial of degree 4. Let* r_1, r_2, r_3, r_4 *denote its zeroes and suppose that* $r_1 + r_2 \in \mathbb{Q}$, *whereas* $r_1 + r_2 \neq r_3 + r_4$. *Prove that also* $r_1 r_2 \in \mathbb{Q}$.

Problem 11.18. $(*)$ *Let* $p(x) \in \mathbb{Z}[x]$ *be a polynomial and let* $m \neq 0$ *be an integer. Define the sequence* (a_n) *by setting* $a_1 = m$ *and* $a_{n+1} = p(a_n)$ *for all* $n \geqslant 1$. *Suppose that* $a_n \neq 0$ *and that* $a_n \mid a_{n+1}$ *for all* $n \geqslant 1$. *Prove that if all* a_i *are distinct, it must hold that* $p(0) = 0$. *Also show that if* $p(0) \neq 0$, *then the sequence eventually becomes periodic with period 1 or 2.*

Problem 11.19. $(*)$ *Let* $p(x) \in \mathbb{Z}[x]$ *be a polynomial with the property that for any integer* $n \geqslant 1$, *it holds that* $p(n) > n$. *Define the sequence* (a_n) *by setting* $a_1 = 1$ *and* $a_{n+1} = p(a_n)$ *for all* $n \geqslant 1$. *Suppose that for any* $m \in \mathbb{N}$, *there exists some* $n \geqslant 1$ *such that* $m \mid a_n$. *Prove that* $p(x) = x + 1$.

Problem 11.20. $(*)$ *Let* $p \in \mathbb{Z}[x]$ *be a polynomial of degree* n *and let* q *be a polynomial of the form* $(p \circ p \circ \ldots \circ p)(x)$. *Prove that there are at most* n *distinct integers* t *such that* $q(t) = t$.

11.4 IMC Problems

Problem 11.21. (2005) *Find all polynomials of degree* n *whose coefficients are a permutation of the numbers* $\{0, 1, \ldots, n\}$ *and all of whose roots are rational numbers.*

Problem 11.22. (2006) *Let* $p, q \in \mathbb{R}[x]$ *be polynomials such that for infinitely many integers* n*, it holds that* $p(n)/q(n)$ *is an integer. Prove that* $q \mid p$.

Problem 11.23. (2007) *Let* $p(x) \in \mathbb{Z}[x]$ *be a polynomial of degree 2. Suppose that* $p(n)$ *is divisible by 5 for every integer* n*. Prove that all coefficients of* p *are divisible by 5.*

Problem 11.24. (2007) *Let* n *be a positive integer and* $a_1, \ldots, a_n$ *be arbitrary integers. Suppose that a function* $f : \mathbb{Z} \to \mathbb{R}$ *satisfies* $\sum_{i=1}^{n} f(k + a_i \ell) = 0$ *whenever* $k, \ell \in \mathbb{Z}$ *and* $\ell \neq 0$*. Prove that* $f = 0$.

Problem 11.25. (2007)$(*)$ *How many nonzero coefficients can a polynomial* $p(x) \in \mathbb{Z}[x]$ *have if* $|p(z)| \leqslant 2$ *for any* $z \in \mathbb{C}$ *with* $|z| = 1$*?*

Problem 11.26. (2008) *Let* $p(x) \in \mathbb{Z}[x]$ *and let* $a_1 < \ldots < a_k$ *be integers. Prove that there exists some* $a \in \mathbb{Z}$ *such that* $p(a_i)$ *divides* $p(a)$ *for all* $1 \leqslant i \leqslant k$*. Does there exist some integer* a *such that the product* $p(a_1) \cdots p(a_k)$ *divides* $p(a)$*?*

Problem 11.27. (2008) *Let* $f(x), g(x) \in \mathbb{Z}[x]$ *be non-constant polynomials such that* $g(x) \mid f(x)$*. Prove that if the polynomial* $f(x) - 2008$ *has at least* 81 *distinct roots, then the degree of* $g(x)$ *is greater than 5.*

Problem 11.28. (2009) *Let* p *be a prime number and let* W *be the smallest subset of* $\mathbb{F}_p[x]$ *such that* $x + 1 \in W$ *and* $x^{p-2} + x^{p-3} + \ldots + x^2 + 2x + 1 \in W$*, and in addition for any* $h_1(x), h_2(x) \in W$*, the remainder of* $h_1(h_2(x))$ *upon division by* $x^p - x$ *is also in* W*. How many polynomials does* W *contain?*

Problem 11.29. (2011) *Let* p *be a prime number. We say a positive integer* n *is* interesting *if*

$$x^n - 1 = (x^p - x + 1)f(x) + pg(x)$$

for some $f, g \in \mathbb{Z}[x]$*. Prove that* $p^p - 1$ *is interesting. For which primes* p *is* $p^p - 1$ *the smallest interesting number?*

Problem 11.30. (2011)$(*)$ *Let* $f(x) \in \mathbb{R}[x]$ *be a polynomial of degree* n*. Suppose that* $\frac{f(k)-f(m)}{k-m}$ *is an integer for all integers* $0 \leqslant k < m \leqslant n$*. Prove that* $a - b$ *divides* $f(a) - f(b)$ *for all pairs of distinct integers* a *and* b.

Problem 11.31. (2012)$(*)$ *Let* $a \in \mathbb{Q}$ *and* $n \in \mathbb{N}$*. Prove that the polynomial* $x^{2^n}(x + a)^{2^n} + 1$ *is irreducible in* $\mathbb{Q}[x]$.

12 Combinatorics II

We leave mere counting behind and explore some widely usable techniques for combinatorial problems.

12.1 Combinatorial techniques

There are some universally applicable techniques in combinatorics which shine as brightly at high school olympiads as they do at the IMC. Many of these will be well known to most, so we only mention the ideas briefly.

Remark 12.1. *(Pigeonhole Principle)* Let $k, n \in \mathbb{N}$. Suppose that more than kn objects are placed into at most n different categories. Then at least one category will contain more than k objects.

This observation is so trivial that it hardly needs justification. However, it is so general that it can still be used to prove strong and non-trivial results. An idea of similar simplicity is examplified by the following.

Remark 12.2. *(Extremal Principle)* Every finite set of real numbers contains both a largest and a smallest member.

What this remark is trying to convey, is that it can sometimes be useful to focus your attention on the largest or smallest number or object in your set, because it typically satisfies some nice properties.

Another common technique is finding an *invariant*. Sometimes a combinatorial problem describes some kind of process and it is asked if a certain situation can or cannot, or must necessarily be reached at some point. Sometimes there are hidden quantities that are secretly always constant, or slightly weaker, are always moving in one direction, even though the situation in the problem itself seems to display little structure. For example, if we have a sequence of numbers defined by $2a_{n+2} = a_{n+1} + a_n$, then without even studying the sequence, we may note that $2a_{n+2} + a_{n+1} = 2a_{n+1} + a_n$; in other words, the sequence $b_n = 2a_{n+1} + a_n$ is constant, so without even analysing the sequence (a_n), we know that the only possible limit could be $(2a_1 + a_0)/3$.

When the problem concerns some kinds of situation that plays out on a square grid, or on some graph, it is often a good idea to consider *colourings* to try to analyse the situation. This can sometimes turn complicated seeming problems into an almost trivial remark. For example, dominoes can't cover a chess board with two opposite corners removed, simply because the number of white and black squares is no longer equal. However, without the colouring, this is not easy to see at all.

12.2 Examples

Example 12.3. Let $n+2$ integers be given. Show that there are two numbers of which either their sum or their difference is divisible by $2n$.

Solution: Consider the $2n$ different residue classes modulo $2n$ and group them together in the following way:

$$\{0\}, \{1, 2n-1\}, \{2, 2n-2\}, \ldots, \{n-1, n+1\}, \{n\}.$$

In total we have made $n+1 < n+2$ groups, so by the pigeonhole principle, at least one of these groups must contain at least two numbers. If any two numbers agree modulo $2n$, we are done. If not, then the group containing at least two numbers must have been of the form $\{k, 2n-k\}$. But then the sum of these two numbers is a multiple of $2n$ and the result is proved. $\triangle$

Example 12.4. Let G be a finite directed graph with the property that for any two vertices v and w there is a path either $v \to w$ or a path $w \to v$. Show that there must be a vertex that can be reached from any other vertex.

Solution: Suppose that such a vertex does not exist. Now let v be the vertex that can be reached from the greatest number of other vertices. By assumption, there exists some vertex w from which we cannot reach v. However, it must then follow that from v we can reach w instead. We claim that w can now be reached from more vertices than v, which would yield a contradiction. Indeed, through v, the vertex w can be reached from all vertices from which v can be reached, and additionally w can be reached from v; thus strictly more. This completes the proof. $\triangle$

Example 12.5. Given a triple of numbers, a step consists of choosing two of these, say a and b, and replacing them by $(a+b)/\sqrt{2}$ and $(a-b)/\sqrt{2}$. Can we turn the triple $(1, \sqrt{2}, 1+\sqrt{2})$ into the triple $(2, 2, \sqrt{2})$ through a sequence of such steps?

Solution: We claim that the sum of the squares of the numbers is invariant throughout the process. Indeed, if we start with the triple (a, b, c) and pick the numbers a and b, we may compute that

$$\begin{aligned}\left(\frac{a+b}{\sqrt{2}}\right)^2 + \left(\frac{a-b}{\sqrt{2}}\right)^2 + c^2 &= \frac{a^2+2ab+b^2}{2} + \frac{a^2-2ab+b^2}{2} + c^2 \\ &= a^2+b^2+c^2.\end{aligned}$$

One checks that for the first given triple the sum of squares equals $6+2\sqrt{2}$, whereas for the latter it equals 10. Therefore, such a sequence of steps cannot exist. $\triangle$

12.3 Exercises

Problem 12.1. *Prove that at every party with at least two people, there exist two guests who know the same number of other people.*

Problem 12.2. *Let* $n \geqslant 1$*. Prove that a square* $2^n \times 2^n$*-grid with one corner removed can be tiled by L-triominos, which are three connected unit squares in an L-shape.*

Problem 12.3. *Consider a sequence of 2023 ones and 2024 zeroes. A move consists of deleting two numbers and writing a new one; if we deleted the same number twice, we must write a 0, otherwise we must write a 1. Do we always end up with the same digit in the end?*

Problem 12.4. *Consider a table with 2023 cards with the numbers* $1, \ldots, 2023$*. A move consists of taking two cards with, say, the numbers* p *and* q*, and replacing them by the single card containing the number* $pq + p + q$*. Which possible values can appear on the final card?*

Problem 12.5. *Three players are sitting at a round table with 3, 4 and 5 coins respectively. Each turn, every player either gives two coins to the person to their right, or one coin to the person to their left. The players win if each player ends up with the same number of coins, and they lose if someone runs out of coins. Can they win the game?*

Problem 12.6. *Consider a* generalised knight*; this chess piece moves on an infinite chess board always* p *squares in one direction and then* q *in a direction perpendicular to it. Prove that such a generalised knight can only return to its starting square in an even number of moves.*

Problem 12.7. *Let* $\alpha > 0$ *be irrational and let* $\epsilon > 0$*. Prove that there exists positive integers* k *and* n *such that* $|k\alpha - n| < \epsilon$*.*

Problem 12.8. *Two players play a game with* $2n$ *cards, numbered 1 to* $2n$*. The cards are shuffled and distributed evenly to the two players. The two players take turns putting cards on the table; if one of the players can make the sum of the cards on the table a multiple of* $2n + 1$*, they win the game. Which player has a winning strategy?*

Problem 12.9. *Let* n *integers be given. Prove that there exists a non-empty subset of these numbers such that their sum is divisible by* n*.*

Problem 12.10. *Let* a_j, b_j, c_j *be integers for all* $1 \leqslant j \leqslant n$ *such that for each* j*, at least one of the numbers* a_j*,* b_j *and* c_j *is odd. Show that there are integers* r*,* s *and* t *such that the expression* $ra_j + sb_j + tc_j$ *is odd for at least* $4n/7$ *different values of* j*.*

Problem 12.11. *The numbers* $1, 2, \ldots, 2025$ *are written on a blackboard. A move consists of replacing two numbers by their absolute difference. Can we reach a situation with only zeroes on the board?*

Problem 12.12. *Let* $m, n \in \mathbb{N}$ *and let* $a_0, \ldots, a_{mn}$ *be distinct real numbers. Show that there exists either an increasing sequence of length* $m + 1$ *or a decreasing sequence of length* $n + 1$*.*

Problem 12.13. *We have a stack containing 1001 stones. A move consists of choosing a stack containing at least 3 stones, tossing one stone out and splitting the stack into two smaller stacks. Can we end up with only stacks containing precisely 3 stones?*

Problem 12.14. *Prove that for any* $n \geqslant 2$*, the polynomial* $(1 + X + X^2)^n \in \mathbb{Z}[X]$ *contains at least one even coefficient.*

Problem 12.15. *On a party with* $n \geqslant 4$ *people, it turns out that for any four guests, we can either find three guests who do all know each other, or three guests who do all not know each other. Prove that we can divide all the guests over two rooms such that any two people from different rooms do not know each other.*

Problem 12.16. *Let* $n \geqslant 1$ *and let* $0 < a_1 < \ldots < a_n$ *be real numbers. Prove that the set*

$$\{\pm a_1 \pm a_2 \pm \ldots \pm a_n\}$$

contains at least $n(n+1)/2$ *elements.*

Problem 12.17. *Let* n *points be given on a circle and draw all possible chords between them. Suppose that no three chords intersect at a common point inside the circle. How many intersections points can we find inside the circle?*

Problem 12.18. *On a sheet of paper, a few positive real numbers are written with the property that the sum of all their pair-wise products is equal to 1. Prove that we can erase one of the numbers such that the sum of the remaining numbers is at most* $\sqrt{2}$.

Problem 12.19. *A group of students satisfies the property that if two students have the same number of friends in the group, then they have no friends in common. Given that at least one pair of students is befriended, prove that there is a student with precisely one friend.*

Problem 12.20. *In a group of* $2n$ *people, every person knows at most* $n-1$ *other people. Prove that we can seat everyone at a round table such that nobody sits next to someone they know.*

Problem 12.21. *Consider any set of* $2n$ *points in the plane, precisely half of which are red and half of which are blue. Prove that we can pair up the red and blue points bijectively in such a way that for any two such pairs, the straight line segments connecting the red point and the blue point of the pair do not intersect.*

Problem 12.22. *In a group of* $3n$ *people it is known that every person has ever hit at most one other person in the group. Prove that we can find a subgroup of* n *people such that nobody has ever hit anyone else.*

Problem 12.23. *Three different schools have* n *students each. It is known that every student has precisely* $n+1$ *friends outside of his own school. Prove that one can find three students, one from each school, such that they are all friends.*

Problem 12.24. *Each of the vertices of an* n*-gon is given a stack of coins. Repeatedly, we choose one vertex with at least two coins and distribute one coin to each of its neighbours. It turns out that after precisely* k *steps, we are precisely back in the original configuration. Show that* $n \mid k$.

Problem 12.25. *A group of* 2024 *people stand around a large circle. Alice starts with the ball and each turn, a person plays the ball to one of their neighbours, or the person directly opposite them. They play 2023 turns such that every person has had the ball exactly once. How many people can be the last one with the ball?*

Problem 12.26. *We are given a grid of* 2023×2023 *light bulbs, all of which are on. If you touch a bulb, itself and all bulbs in the same row and column will change their state. What is the smallest number of light bulbs we have to touch to turn all bulbs off?*

Problem 12.27. *Let* $k \geqslant 1$. *We are given a sequence of* $4k$ *coins, of which precisely* $2k$ *are heads and* $2k$ *are tails. A move consists of swapping any set of adjacent heads with an equal number of adjacent tails. What is the smallest number* n *such that we can be sure to be able to change any such sequence into the sequence starting with* $2k$ *heads in at most* n *moves?*

Problem 12.28. *A collection of subsets from* $\{1, 2, \ldots, n\}$ *has the property that any two subsets from the collection have non-empty intersection. How large than such a collection be?*

Problem 12.29. *Let* $n + 1$ *numbers from the set* $\{1, 2, \ldots, 2n\}$ *be given. Prove that one of these numbers is divisible by another.*

Problem 12.30. *Can a* 2023×2023 *board that misses one of its corner pieces be covered by an equal number of horizontal and vertical dominoes?*

Problem 12.31. *On each square of a* 9×9 *grid sits an ant. All at once, every ant moves to a square diagonally adjacent to its current square. What is the minimal number of squares that are now empty?*

Problem 12.32. *Consider a set with* 2023 *positive integers, none of them containing a prime factor exceeding* 23. *Prove that this set contains four numbers whose product is a fourth power.*

Problem 12.33. *Consider an* $n \times n$ *board, some of whose squares are considered sick. If a healthy square borders at least two sick squares, it will become sick too. At the end of the epidemic, the whole board turns out to be sick. How many squares must have been initially sick?*

Problem 12.34. *Consider a pawn situated in the origin* $(0, 0)$ *of the plane. A move consists of removing a pawn from a point* (a, b) *and placing a pawn on the points* $(a + 1, b)$ *and* $(a, b + 1)$, *provided these squares are empty. Prove that there will always be at least one pawn a distance at most* $\sqrt{5}$ *from the origin.*

Problem 12.35. *A chess player prepares himself for a tournament during the coming* 77 *days. He will play at least one game each day, but at most* 132 *games in total. Prove that there is a sequence of consecutive days during which the chess player plays precisely 21 games.*

Problem 12.36. *There is a coin placed heads up on every square of an* $m \times n$ *board. A step consists of choosing a coin that is heads up, removing that coin and then flipping over all its neighbours. For which values of* m *and* n *is it possible to remove all coins from the board?*

Problem 12.37. *Let* $n_1 < \ldots < n_{2000} < 10^{100}$ *be given. Prove that we can split these numbers into two disjoint sets of equal size of which the sum of their elements and the sum of the squares of their elements both agree.*

Problem 12.38. *Let* n *be an odd positive integer and colour an* $n \times n$ *board in a chess-board pattern in such a way that all four corners are black. For which* n *is it possible to place down L-triomino's on the board in such a way that all the black squares are covered?*

Problem 12.39. *Let nine lines be given that divide a given square into two quadrilaterals whose areas relate as* $2:3$. *Prove that three of these lines must intersect in a single point.*

Problem 12.40. *For a partition* π *of the set* $S = \{1, 2, \ldots, 9\}$, *let* $\pi(x)$ *denote the size of the set of* π *containing* x. *Prove that for any two such partitions* π *and* π', *there exist* $x, y \in S$ *with* $\pi(x) = \pi(y)$ *and* $\pi'(x) = \pi'(y)$.

Problem 12.41. *Let* (x_n) *be the sequence defined by* $x_n = n$ *for all* $n \leqslant 2023$ *and* $x_{n+1} = x_n + x_{n-2023}$ *for all* $n \geqslant 2023$. *Prove that this sequence contains 2022 subsequent terms which are all divisible by 2023.*

Problem 12.42. *We construct a sequence that starts with* $1, 0, 1, 0, 1, 0, \ldots$ *and the next term is always given by the value mod 10 of the sum of the six before it. Prove that* $0, 1, 0, 1, 0, 1$ *never appears in the sequence.*

Problem 12.43. *In a group of people, every person has at most three enemies. Prove that we can divide this group over two rooms such that every person has at most one of its enemies with them in the same room.*

Problem 12.44. *Let* $k \in \mathbb{N}$ *and consider a regular 12-gon and call one of its vertices special. At each vertex we place a coin, such that only the coin at the special vertex is tails up. A move consists of selecting* k *adjacent coins and flipping all of them around. Can we reach a situation in which one of the neighbours of the special vertex is the only one with a coin that is tails up if* $k = 3$? *What if* $k = 4$? *And* $k = 5$?

Problem 12.45. *Let a rectangular board be covered completely by* 1×4 *and* 2×2 *tiles. Suppose that we remove one* 2×2 *tile and add one* 1×4 *tile back in its place. Is it possible for us to still cover the original board?*

Problem 12.46. *A pawn on a chess board can move either a square to the right, a square up, or a square to the bottom-left. Can it make a journey across the board, visiting all squares exactly once, and ending up one square to the right of its initial square?*

Problem 12.47. *Let* N *be a positive integer. Can a knight make a journey across all squares of a* $4 \times N$*-boards, visiting each square exactly once, and return back to its original square?*

Problem 12.48. *At 7 of the vertices of a cube we write a 0 and at the last vertex we write a 1. A move consists of choosing a face of the cube and adding 1 to each vertex bounding the face. Can we make the numbers at all vertices equal? Can we make all of them divisible by 3?*

Problem 12.49. *On every point* $(a, b) \in \mathbb{Z}^2$ *with* $b < 0$ *we have a pawn. A move consists of letting a pawn jump over another pawn (like in checkers) and to remove the pawn we just jumped over. On which points in* $\mathbb{Z}^2$ *can be get a pawn?*

Problem 12.50. *On an infinite chessboard, we have an* $n \times n$ *square of pawns. A move consists of letting a pawn jump over another pawn (like in checkers) and to remove the pawn we just jumped over. For which values of* n *can we end up with only a single pawn left on the board?*

Problem 12.51. *A group of n players sit around a table, each with one coin. One person starts by giving one coin to the person to their right. Then that person hands two coins to the person to their right. Then that person gives passes on one coin again, and the next two again, etcetera. Each time someone runs out of money, they leave the table. Find infinitely many values of n for which in the end, one person ends up with all the coins.*

Problem 12.52. *Consider a polytope with at least five faces such that in every corner, precisely 3 sides meet. Two players take turns colouring one of the faces in their favourite colour. Whoever first manages to colour all three faces that meet in a single corner with their colour, wins. Which player has a winning strategy?*

Problem 12.53. *Two players take turns writing zeroes or ones in an empty 3×3 matrix. Player 1 wins if the resulting matrix is invertible; otherwise player 2 wins. Which player has a winning strategy? What if they can write any number during their turn instead?*

Problem 12.54. $(*)$ *Let V be a finite set of points in the plane, not all on a line. Prove that there exists a line that passes through precisely two points of V.*

Problem 12.55. $(*)$ *Let $m, n > 1$ be integers. Alice divides the set of numbers $\{1, 2, \ldots, 2m\}$ into m pairs. Then, Bob chooses one number from each pair and computes their total sum. Prove that Alice can choose the pairs in such a way that Bob cannot hope to get n as his final answer.*

Problem 12.56. $(*)$ *We are given a finite graph in which every vertex can either be coloured red or blue. A move consists of choosing a vertex and changing its colour and the colour of its neighbours. Suppose that all vertices start red. Prove that it is possible to turn all vertices blue.*

Problem 12.57. $(*)$ *Let $f : \mathbb{N} \times \mathbb{N} \to \{\pm 1\}$ be a function. Prove that there exists an infinite subset $V \subset \mathbb{N}$ such that f is constant on $V \times V$.*

Problem 12.58. $(*)$ *We are given a regular pentagon and we associate to each vertex an integer such that their sum is positive. If x, y and z are adjacent, we may replace these numbers by $x + y$, $-y$ and $y + z$ respectively. If we continue doing this, will we inevitably end up with a situation in which all numbers are non-negative?*

Problem 12.59. $(*)$ *Given a simple graph, we call a set S of vertices* covering *if for every vertex, either itself or precisely one of its neighbours is in S. Prove that any two coverings, if they exist, must have the same size.*

12.4 IMC Problems

Problem 12.60. (1997) *Suppose that* F *is a family of finite subsets of* $\mathbb{N}$ *with the property that for any* $A, B \in F$ *we have that* $A \cap B \neq \varnothing$. *Is it true that there must be some finite* $Y \subset \mathbb{N}$ *such that for any* $A, B \in F$ *we have that* $A \cap B \cap Y \neq \varnothing$? *What if we additionally suppose that all members of* F *have the same size?*

Problem 12.61. (1997) *Let* X *be a set and let* $f : X \to X$ *be an bijective function. Prove that there exists functions* $g_1, g_2 : X \to X$ *such that* $f = g_1 \circ g_2$ *and* $g_1 \circ g_1 = \mathrm{id} = g_2 \circ g_2$.

Problem 12.62. (1999) *Suppose that* $2n$ *points of an* $n \times n$ *grid are marked. Show that for some* $k > 1$, *one can select* $2k$ *distinct marked points, say* $a_1, \ldots, a_{2k}$, *such that* a_{2i-1} *and* a_{2i} *are in the same row for all* $1 \leqslant i \leqslant k$ *and* a_{2i} *and* a_{2i+1} *are in the same column for all* $1 \leqslant i < k$, *and additionally* a_{2k} *and* a_1 *are in the same column.*

Problem 12.63. (1999)$(*)$ *Let S be the collection of all words using only the letters* x, y *and z. Consider an equivalence relation on S given by the relations* $uu \sim u$ *for all* $u \in S$, *and if* $v \sim w$, *then also* $uv \sim uw$ *and* $vu \sim wu$ *for all* $u, v, w \in S$. *Prove that every word in S is equivalent to a word of length at most 8.*

Problem 12.64. (2000) *Show that the unit square can be partitioned into* n *smaller squares for all* $n \geqslant 6$. *Further, show for any* $d \geqslant 2$ *that there is a constant* $N(d)$ *such that, whenever* $n \geqslant N(d)$, *a* d*-dimensional unit cube can be partitioned into* n *smaller cubes.*

Problem 12.65. (2002) *A total of 200 students participated with a math olympiad that featured 6 problems. Every problem was solved by at least 120 students. Prove that there are two students who together solved all problems.*

Problem 12.66. (2003)$(*)$ *Find all positive integers* n *for which there exists a family* $\mathcal{F}$ *of three-element subsets of* $S = \{1, 2, \ldots, n\}$ *satisfying that for any two distinct* $a, b \in S$, *there exists exactly one* $A \in \mathcal{F}$ *containing both* a *and* b, *and further satisfying that for any* $a, b, c, x, y, z \in S$ *such that* $\{a, b, x\}, \{a, c, y\}, \{b, c, z\} \in \mathcal{F}$, *then also* $\{x, y, z\} \in \mathcal{F}$.

Problem 12.67. (2004) *Let* $k \geqslant 2$ *and let* X *be a set of* $\binom{2k-4}{k-2} + 1$ *real numbers. Prove that there exists a monotone sequence* $x_1, \ldots, x_k$ *of elements of* X *such that* $|x_{i+1} - x_1| \geqslant 2|x_i - x_1|$ *for all* $1 < i < k$.

Problem 12.68. (2006) *Let* V *be a convex polygon with* n *vertices. Prove that if* n *is divisible by* 3, *then* V *can be triangulated so that each vertex of* V *is the vertex of an odd number of triangles. Further show that if* n *is not divisible by 3, then* V *can be triangulated so that there are exactly two vertices are are the vertices of an even number of triangles.*

Problem 12.69. (2008)$(*)$ *We say a triple* (a_1, a_2, a_3) *of non-negative reals is* better *than another triple* (b_1, b_2, b_3) *if* two out of the three *inequalities* $a_1 > b_1$, $a_2 > b_2$ *and* $a_3 > b_3$ *are satisfied. We call a triple* (x, y, z) *of non-negative reals* special *if* $x + y + z = 1$. *Find all natural numbers* n *for which there is a set* S *of* n special *triples such that for any given* special *triple, we can find at least one* better *triple in* S.

Problem 12.70. (2009) *In a town, every two residents who are not friends have a friend in common, and no one is a friend of everyone else. Let us number the residents from* 1 *to* n *and let* a_i *be the number of friends of the* i*-th resident. Suppose that* $\sum_{i=1}^{n} a_i^2 = n^2 - n$. *Let* $k \geqslant 3$ *be the smallest number of residents who can be seated at a round table in such a way that any two neighbors are friends. Determine all possible values of* k.

Problem 12.71. (2011) *An alien race has three genders: male, female, and emale. A married triple consists of three persons, one from each gender, who all like each other. Any person is allowed to belong to at most one married triple. A special feature of this race is that feelings are always mutual; if* x *likes* y, *then* y *likes* x *too. The race is sending an expedition to colonize a planet. The expedition has* n *males,* n *females, and* n *emales. It is known that every expedition member likes at least* k *persons of each of the two other genders. The problem is to create as many married triples as possible to produce healthy offspring so the colony could grow and prosper. Show that if* n *is even and* $2k = n$, *then it might be impossible to create even one married triple. Also show that if* $4k \geqslant 3n$, *then it is always possible to create* n *disjoint married triples, thus marrying all of the expedition members.*

Problem 12.72. (2013) *There are* $2n \geqslant 4$ *students in a school. Each week,* n *students go on a trip. After several trips, the following condition was fulfilled: every two students were together on at least one trip. What is the minimum number of trips needed for this to happen?*

Problem 12.73. (2016)($*$) *Let* $n \geqslant k$ *be positive integers and let* $\mathcal{F}$ *be a family of finite sets with the properties that* $\mathcal{F}$ *contains at least* $\binom{n}{k} + 1$ *distinct sets of size* k, *and such that for any two sets* $A, B \in \mathcal{F}$, *their union* $A \cap B$ *also belongs to* $\mathcal{F}$. *Prove that* $\mathcal{F}$ *contains at least three sets with at least* n *elements.*

Problem 12.74. (2017) *There are* n *people in a city and each of them has exactly 1000 friends; fortunately, friendship is symmetric. Prove that it is possible to select a group* S *of people such that at least* $n/2017$ *persons in* S *have exactly two friends in* S.

Problem 12.75. (2022)($*$) *Let* $n > 3$ *be an integer and let* Ω *denote the set of all triples of distinct elements of* $\{1, 2, \dots, n\}$. *Let* m *denote the minimal number of colours which suffice to colour* Ω *so that whenever* $1 \leqslant a < b < c < d \leqslant n$, *the triples* $\{a, b, c\}$ *and* $\{b, c, d\}$ *have different colours. Prove that*

$$\frac{1}{100} \log \log n \leqslant m \leqslant 100 \log \log n.$$

Problem 12.76. (2023)($*$) *We say that a real number* V *is* good *if there exist two closed convex subsets* X *and* Y *of the unit cube in* $\mathbb{R}^3$, *with volume* V *each, such that for each of the three coordinate planes (that is, the planes spanned by two of the three coordinate axes), the projections of* X *and* Y *onto that plane are disjoint. Find* $\sup\{V \mid V \text{ is good}\}$.