Number Theory Exercises

Look at the exercises below (mostly from here). Try to solve as many as possible before our meeting. Unless stated otherwise, all variables here take integer values.

1.
(a) Suppose $a^2+b^2=c^2$. Show that 3 divides ab.
(b) Suppose $a^2+b^2=c^2$. What is the largest k such that k must divide abc?
(c) Show 2^x+25^y cannot be a perfect square for positive integers x and y.

2. The numbers 2^{2024} and 5^{2024} are written in decimal notation, one after the other. How many digits are written altogether?

3. Show there exist 2024 consecutive positive integers, each one divisible by the cube of some integer greater than one.

4. Prove that if $n>1$, then n does not divide 2^n-1.

5. How many primes, when written in decimal notation, have digits that are alternating 1's and 0's?

6. Prove that no number of the form 10^{3n+1} is the sum of two perfect cubes.

7. [2018 A-1] Find all ordered pairs (a,b) of positive integers for which

$$\frac{1}{a}+\frac{1}{b}=\frac{3}{2018}.$$

8. [2017 A-1] Let S be the smallest set of positive integers such that (a) 2 is in S,
(b) n is in S whenever n^2 is in S, and
(c) $(n+5)^2$ is in S whenever n is in S.
Which positive integers are not in S?

9. [2005 A-1] Show that every positive integer is a sum of one or more numbers of the form 2^r3^s, where r and s are non-negative integers and no summand divides another. (For example, 23=9+8+6.)