

PUTNAM PROBLEMS
PROBABILITY AND STATISTICS

2017-A-5. Each of the integers from 1 to n is written on a separate card, and then the cards are combined into a deck and shuffled. Three players, A , B , and C , take turns in the order A, B, C, A, \dots choosing one card at random from the deck. (Each card in the deck is equally likely to be chosen.) After a card is chosen, that card and all higher-numbered cards are removed from the deck, and the remaining cards are reshuffled before the next turn. Play continues until one of the three players wins the game by drawing the card numbered 1.

Show that for each of the three players, there are arbitrarily large values of n for which that player has the highest probability among the three players of winning the game.

2016-B-4. Let A be a $2n \times 2n$ matrix, with entries chosen at random. Each entry is chosen to be 0 or 1, each with probability $1/2$. Find the expected value of $\det(A - A')$ as a function of n , where A' is the transpose of A .

2014-A-4. Suppose X is a random variable that takes on only nonnegative integers values, with $E[X] = 1$, $E[X^2] = 2$, and $E[X^3] = 5$. (Here $E[Y]$ denotes the expectation of the random variable Y .) Determine the smallest possible value of the probability of the event $X = 0$.

2011-A-6. Let G be an abelian group with n elements, and let

$$\{g_1 = e, g_2, \dots, g_k\} \subseteq G$$

be a (not necessarily minimal) set of distinct generators of G . A special die, which randomly selects one of the elements g_1, g_2, \dots, g_k with equal probability, is rolled m times and the selected elements are multiplied to produce an element $g \in G$.

Prove that there exists a real number $b \in (0, 1)$ such that

$$\lim_{m \rightarrow \infty} \frac{1}{b^{2m}} \sum_{x \in G} \left(\text{Prob}(g = x) - \frac{1}{n} \right)^2$$

is positive and finite.

2007-A-3. Let k be a positive integer. Suppose that the integers $1, 2, 3, \dots, 3k + 1$ are written down in random order. What is the probability that at no time during the process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

2006-A-6. Four points are chosen uniformly and independently at random in the interior of a given circle. Find the probability that they are the vertices of a convex quadrilateral.

2005-A-6. Let n be given, $n \geq 4$, and suppose that P_1, P_2, \dots, P_n are n randomly, independently and uniformly, chosen points on a circle. Consider the convex n -gon whose vertices are the P_i . What is the probability that at one of the vertex angles of this polygon is acute?

2004-A-5. An $m \times n$ checkerboard is coloured randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, p and q , are in the same connected monochromatic region if there is a sequence of squares, all of the same colour, starting at p and ending at q , in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

2002-B-1. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability that she hits exactly 50 of her first 100 shots?

2002-B-4. An integer n , unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n in an **odd** number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you **must** guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that your chance of winning is greater than $2/3$.

2001-A-2. You have coins C_1, C_2, \dots, C_n . For each k , coin C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n .

1995-A-6. Suppose that each of n people writes down the numbers 1, 2, 3 in random order in one column of a $3 \times n$ matrix, with all orders equally likely and with the orders for different columns independent of each other. Let the row sums a, b, c of the resulting matrix be rearranged (if necessary) so that $a \leq b \leq c$. Show that, for some $n \geq 1995$, it is at least four times as likely that both $b = a + 1$ and $c = a + 2$ as that $a = b = c$.

1993-B-2. Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players A and B . Beginning with A , the players take turns discarding one of their remaining cards and announcing the number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game. If we assume optimal strategy by both A and B , what is the probability that A wins?

1993-B-3. Two real numbers x and y are chosen at random in the interval $(0, 1)$ with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

1992-A-6. Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

1989-A-4. If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is α ? (An honest coin is one for which the probability of heads and the probability of tails are both $\frac{1}{2}$. A game is finite if with probability 1 it must end in a finite number of moves.)

1989-B-6. Let (x_1, x_2, \dots, x_n) be a point chosen at random from the n -dimensional region defined by $0 < x_1 < x_2 < \dots < x_n < 1$. Let f be a continuous function on $[0, 1]$ with $f(1) = 0$. Set $x_0 = 0$ and $x_{n+1} = 1$. Show that the expected value for the Riemann sum

$$\sum_{i=1}^n (x_{i+1} - x_i) f(x_{i+1})$$

is $\int_0^1 f(t)P(t)dt$ where P is a polynomial of degree n , independent of f , with $0 \leq P(t) \leq 1$ for $0 \leq t \leq 1$.