PUTNAM PROBLEMS

MATRICES, DETERMINANTS AND LINEAR ALGEBRA

2018-A-2. Let S_1, S_2, …, S_{2^n-1} be the nonempty subsets of $\{1, 2, \ldots, n\}$ in some order, and let M be the $(2^n - 1) \times (2^n - 1)$ matrix whose (i, j) entry is

$$m_{ij} = \begin{cases} 0 & \text{if } S_i \cap S_j = \emptyset; \\ 1 & \text{otherwise.} \end{cases}$$

Calculate the determinant of M.

2018-B-1. let $\mathfrak{P}$ be the set of vectors defined by

$$\mathfrak{P} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| \quad 0 \le a \le 2, 0 \le b \le 100, \quad \text{and} \quad a, b, \in \mathbf{Z} \right\}.$$

Find all $\mathbf{v} \in \mathfrak{P}$ such that $\mathfrak{P} \setminus \{\mathbf{v}\}$ obtained by omitting vector $\mathbf{v}$ from $\mathfrak{P}$ can be partitioned into two sets of equal size and equal sum.

2016-B-4. Let A be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability 1/2. Find the expected value of $\det(A - A')$ (as a function of n), where A' is the transpose of A.

2015-A-6. Let n be a positive integer. Suppose that A, B, and M are $n \times n$ matrices with real entries such that $AM = MB$, and such that A and B have the same characteristic polynomial. Prove that $\det(A - MX) = \det(B - XM)$ for every $n \times n$ matrix X with real entries.

2015-B-3. Let S be the set of all 2×2 matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

whose entries a, b, c, d (in that order) form an arithmetic progression. Find all matrices M in S for which there is some integer $k > 1$ such that M^k is also in S.

2014-A-2. Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is

$$\frac{1}{\min(i, j)}$$

for $1 \le i, j \le n$. Compute $\det(A)$.

2014-A-6. Let n be a positive integer. What is the largest k for which there exist $n \times n$ matrices $M_1, \ldots, M_k$ and $N_1, \ldots, N_k$ with real entries such that, for all i and j, the matrix product $M_i N_j$ has a zero entry somewhere on its main diagonal if and only if $i \ne j$?

2014-B-3. Let A be an $m \times n$ matrix with rational entries. Suppose that there are at least $m + n$ distinct prime numbers among the absolute values of the entries of A. Show that the rank of A is at least 2.

2014-B-5. In the 75th Annual Putnam Games, participants compete at mathematical games. Patniss and Keeta play a game in which they take turns choosing an element from the group of invertible $n \times n$ matrices with entries in the field $\mathbf{Z}/p\mathbf{Z}$ of integers modulo p, where n is a fixed positive integer and p is a fixed prime number. The rules of the game are:

1. A player cannot choose an element that has been chosen by either player on any previous term.

2. A player can only choose an element that commutes with all previously chosen elements.

3. A player who cannot choose an element on his/her turn loses the game.

Patniss takes the first turn. Which player has a winning strategy? (Your answer may depend on n and p.)

2012-A-5. Let $\mathbf{F}_p$ denote the field of integers modulo a prime p, and let n be a positive integer. Let v be a field vector in $\mathbf{F}_p^n$ and let M be an $n \times n$ matrix with entries in $\mathbf{F}_p$, and define $G : \mathbf{F}_p^n \to \mathbf{F}_p^n$ by $G(x) = v + Mx$. Let $G^{(k)}$ denote the $k-$fold composition of G with itself, that is $G*(1)(x) = G(x)$ and $G^{(k+1)}(x) = G(G^{(k)}(x))$. Determine all pairs p, n for which there exist v and M such that the p^n vectors $G^{(k)}(0)$, $k = 1, 2, \cdots, p^n$ are distinct.

2011-A-4. For which positive integers n is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, while every dot product of two different rows is odd?

2011-B-4. In a tournament, 2011 players meet 2011 times to play a multiplayer game. Each game is played by all 2011 players together and each ends with each of the players either winning or losing. The standings are kept in two 2011×2011 matrices, $\mathbf{T} = (T_{hk})$ and $\mathbf{W} = (W_{hk})$. Initially, $\mathbf{T} = \mathbf{W} = \mathbf{0}$. After every game, for every (h, k) (including $h = k$), if players h and k tied (that is, both won or both lost), then the entry T_{hk} is increased by 1, while if player h won and player k lost, the entry W_{hk} is increased by 1 and W_{kh} is decreased by 1.

Prove that, at the end of the tournament, $\det(\mathbf{T} + i\mathbf{W})$ is a non-negative integer divisible by 2^{2010}.

2010-B-6. Let A be an $n \times n$ matrix of real numbers for some $n \geq 1$. For each positive integer k, let $A^{[k]}$ be the matrix obtained by raising each entry to the kth ppower. Show that if $A^k = A^{[k]}$ for $k = 1, 2, \ldots, n+1$, then $A^k = A^{[k]}$ for all $k \geq 1$.

2009-A-3. Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1$, $\cos 2$, $\ldots$, $\cos n^2$. (For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

The argument of cos is always in radians, not degrees.) Evaluate $\lim_{n\to\infty} d_n$.

2009-B-4. Say that a polynomial with real coefficients in two variables, x, y, is *balanced* if the average value of the polynomial on each circle centered at the origin is 0. The balanced polynomials of degree at most 2009 form a vector space V over $\mathbf{R}$. Find the dimension of V.

2008-A-2. Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if its is zero. Which player has a winning strategy?

2006-B-4. Let Z denote the set of points in $\mathbf{R}^n$ whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are vertices of a hypercube in $\mathbf{R}^n$.) Given a vector subspace V of $\mathbf{R}^n$, let $Z(V)$ denote the number of members of Z that lie in V. Let k be given, $0 \leq k \leq n$. Find the maximum, over all vector subspaces $V \subseteq \mathbf{R}^n$ of dimension k, of the number of points in $V \cap Z$.

2005-A-4. Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

2004-A-3. Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

2003-B-1. Do there exist polynomials $a(x)$, $b(x)$, $c(y)$, $d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

2002-A-4. In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

1999-B-5. For an integer $n \geq 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where I is the $n \times n$ identity matrix and $A = (a_{jk})$ has entries $a_{jk} = \cos(j\theta + k\theta)$ for all j, k.

1996-B-4. For any square matrix A, we can define $\sin A$ by the usual power series

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1} \quad .$$

Prove or disprove: There exists a 2×2 matrix A with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix} \quad .$$

1995-B-3. To each positive integer with n^2 decimal digits we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate $\det \begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix} = 50$. Find as a function of n, the sum of all the determinants associated with n^2-digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.)

1994-A-4. Let A and B be 2×2 matrices with integer entries such that A, $A + B$, $A + 2B$, $A + 3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that $A + 5B$ is invertible and that its inverse has integer entries.

1994-B-4. For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \qquad \text{and} \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad .$$

Show that $\lim_{n\to\infty} d_n = \infty$.

1992-B-5. Let D_n denote the value of the $(n-1) \times (n-1)$ determinant

$$\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ . & . & . & . & \cdots & . \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{vmatrix}$$

Is the set

$$\{\frac{D_n}{n!} : n \geq 2$$

bounded?

1992-B-6. Let M be a set of real $n \times n$ matrices such that
(i) $I \in M$, where I is the $n \times n$ identity matrix;
(ii) if $A \in M$ and $B \in M$, then either $AB \in M$ and $-AB \in M$, but not both;
(iii) if $A \in M$ and $B \in M$, then either $AB = BA$ or $AB = -BA$;
(iv) if $A \in M$ and $A \neq I$, then there is at least one $B \in M$ such that $AB = -BA$.
Prove that M contains at most n^2 matrices.

1991-A-2. Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

1990-A-5. If A and B are square matrices of the same size such that $ABAB = 0$, does it follow that $BABA = 0$?

1986-A-4. A *transversal* of an $n \times n$ matrix A consists of n entries of A, no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices A satisfying the following two conditions:
(a) Each entry $\alpha_{i,j}$ of A is in the set $\{-1, 0, 1\}$.
(b) The sum of the n entries of a transversal is the same for all transversals of A.
An example of such a matrix A is

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad .$$

Determine with proof a formula for $f(n)$ of the form

$$f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4 \quad .$$

where the a_i's and b_i's are rational numbers.

1986-B-6. Suppose that A, B, C, D are $n \times n$ matrices with entries in a field F, satisfying the conditions that AB^t and CD^t are symmetric and $AD^t - BC^t = I$. Here I is the $n \times n$ identity matrix, M^t is the transpose of M. Prove that $A^tD - C^tB = I$.

1985-B-6. Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \leq i \leq r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^r \text{tr } (M_i) = 0$, where tr (A) denotes the trace of the matrix A. Prove that $\sum_{i=1}^r M_i$ is the $n \times n$ zero matrix.