Polynomial Exercises

Look at the problems below (from Sasha) and try to solve as many as possible before our meeting. Unless stated otherwise, you may assume all polynomial coefficients are real numbers.

1. For numbers $x_1 < x_2 < \cdots < x_n$ and a number x different from them, prove the identity

$$\frac{1}{(x-x_1)(x-x_2)\dots(x-x_n)} = \sum_{i=1}^{n} \frac{1}{x-x_i} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{x_i - x_j}.$$

2. Polynomial Division with Remainder. Let $P(x)$ and $Q(x)$ be polynomials, with $Q(x)$ not identically equal to zero. Prove that there exist polynomials $T(x)$ and $R(x)$ such that

$$P(x) = Q(x)T(x) + R(x),$$

and $\deg R(x) < \deg Q(x)$; and show that $T(x)$ and $R(x)$ are uniquely determined in this case.

3. Bezout's Theorem. Prove that the remainder of the division of the polynomial $P(x)$ by $x-c$ is equal to $P(c)$.

4. Prove that a polynomial of degree n has no more than n roots.

5. Descartes' Rule of Signs. Prove that the number of positive roots of the polynomial

$$f(x) = a_n x^n + \cdots + a_1 x + a_0$$

does not exceed the number of sign changes in the sequence $a_n, \dots, a_1, a_0$.

6. Find all polynomials $P(x)$ such that for all x

$$(x+1)P(x) = (x-10)P(x+1)$$

7. Find all polynomials P for which there is a positive integer n such that for all x we have

$$P\left(x - \frac{1}{n}\right) + P\left(x + \frac{1}{n}\right) = 2\,P(x)$$

8. Let a, b, c, d, e be integers such that $a+b+c+d+e$ and $a^2+b^2+c^2+d^2+e^2$ are both divisible by some odd natural number n. Prove that $a^5+b^5+c^5+d^5+e^5-5abcde$ is also divisible by n.

9. Find all polynomials P with integer coefficients such that for all x we have

$$P'(P(x)) = P(P'(x))$$