Pigeon-Hole Putnam Problems

Below are Putnam problems with solutions using the pigeon-hole principle. As usual, there are problems on the first page, hints on the second page, and solutions after that.

1 Problems

Look over the problems below. Try to identify one or more problems where you have some idea of how to start. If you make any progress, try to write it down!

1993 A–4 Let $x_1, x_2, \ldots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, y_2, \ldots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.

2000 B–6 Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \pm 1, \ldots, \pm 1)$ in n-dimensional space with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

2002 A–2 Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

2006 B–2 Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

2010 B–3 There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving *exactly* i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

2 Hints

You don't get hints on a real exam, but these kinds of ideas may help with similar problems. Look these hints if you like, and see if you can make any further progress.

1993 A–4 Look at prefix sums, and how they "merge" together.

2000 B–6 A strong pigeon-hole argument.

2002 A–2 A simple argument, start with a great circle.

2006 B–2 Look at partial sums mod 1.

2010 B–3 Make heavy use of B_1.

The next page has solutions, don't continue until you want to see them!

3 Solutions

1993 A–4 (from prase.cz) Let $m = 19$, $n = 93$. So $1 \le x_i \le n$ for $1 \le i \le m$, and $1 \le y_j \le m$ for $1 \le j \le n$.

Define the prefix sums $a_h = \sum_{i=1}^{h} x_i$, $b_k = \sum_{j=1}^{k} y_j$. We suppose $a_m \le b_n$ (if not, reverse the roles of x, a and y, b in the following). For each h we have $a_h \le a_m \le b_n$, so we may define $f(h)$ as the smallest k such that $b_k \ge a_h$. Consider this sequence of m numbers: $b_{f(1)} - a_1, b_{f(2)} - a_2, \ldots, b_{f(m)} - a_m$. If any is zero we are done, so we assume they are all positive. Each number is less than m, because $b_{f(i)} \ge a_i + m$ would imply $b_{f(i)-1} \ge a_i$, contradicting the minimality of $f(i)$. We have a sequence of m numbers drawn from $\{1, 2, \ldots, m-1\}$, so two of them are equal: $b_{f(r)} - a_r = b_{f(s)} - a_s$ with $r < s$. Then $b_{f(s)} - b_{f(r)} = a_s - a_r$. Note the left-hand-side is a sum of $y_j's$, and the right-hand-side is a sum of $x_i's$.

2000 B–6 For each point P in B, let S_P be the set of points with all coordinates equal to ± 1 which differ from P in exactly one coordinate. Since there are more than $2^{n+1}/n$ points in B, and each S_P has n elements, the cardinalities of the sets S_P add up to more than 2^{n+1}, which is to say, more than twice the total number of points. By the pigeonhole principle, there must be a point in three of the sets, say S_P, S_Q, S_R. But then any two of P, Q, R differ in exactly two coordinates, so PQR is an equilateral triangle, as desired.

2002 A–2 Draw a great circle through two of the points. There are two closed hemispheres with this great circle as boundary, and each of the other three points lies in one of them. By the pigeonhole principle, two of those three points lie in the same hemisphere, and that hemisphere thus contains four of the five given points.

2006 B–2 Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. For $i = 0, \ldots, n$, put $s_i = x_1 + \cdots + x_i$ (so that $s_0 = 0$). Sort the numbers $\{s_0\}, \ldots, \{s_n\}$ into ascending order, and call the result $t_0, \ldots, t_n$. Since $0 = t_0 \le \cdots \le t_n < 1$, the differences

$$t_1 - t_0, \ldots, t_n - t_{n-1}, 1 - t_n$$

are nonnegative and add up to 1. Hence (as in the pigeonhole principle) one of these differences is no more than $1/(n+1)$; if it is anything other than $1 - t_n$, it equals $\pm(\{s_i\} - \{s_j\})$ for some $0 \le i < j \le n$. Put $S = \{x_{i+1}, \ldots, x_j\}$ and $m = \lfloor s_i \rfloor - \lfloor s_j \rfloor$; then

$$\begin{aligned} \left| m + \sum_{s \in S} s \right| &= |m + s_j - s_i| \\ &= |\{s_j\} - \{s_i\}| \\ &\le \frac{1}{n+1}, \end{aligned}$$

as desired. In case $1 - t_n \le 1/(n+1)$, we take $S = \{x_1, \ldots, x_n\}$ and $m = -\lceil s_n \rceil$, and again obtain the desired conclusion.

2010 B–3 It is possible if and only if $n \ge 1005$. Since

$$1 + \cdots + 2009 = \frac{2009 \times 2010}{2} = 2010 \times 1004.5,$$

for $n \le 1004$, we can start with an initial distribution in which each box B_i starts with at most $i-1$ balls (so in particular B_1 is empty). From such a distribution, no moves are possible, so we cannot reach the desired final distribution.

Suppose now that $n \ge 1005$. By the pigeonhole principle, at any time, there exists at least one index i for which the box B_i contains at least i balls. We will describe any such index as being *eligible*. The following sequence of operations then has the desired effect.

(a) Find the largest eligible index i. If $i = 1$, proceed to (b). Otherwise, move i balls from B_i to B_1, then repeat (a).

(b) At this point, only the index $i = 1$ can be eligible (so it must be). Find the largest index j for which B_j is nonempty. If $j = 1$, proceed to (c). Otherwise, move 1 ball from B_1 to B_j; in case this makes j eligible, move j balls from B_j to B_1. Then repeat (b).

(c) At this point, all of the balls are in B_1. For $i = 2, \dots, 2010$, move one ball from B_1 to B_i n times.

After these operations, we have the desired distribution.