# Pigeon Hole Exercises

Look at the problems below (from Sasha) and try to solve as many as possible before our meeting.

**1.** (Pigeonhole principle) If $n+1$ objects are arranged in $n$ places, there must be at least two objects in the same place.

**2.** (Pigeonhole principle, strong version) If $n$ objects are arranged in $k$ places, there are at least $\lceil n/k \rceil$ objects in the same place.

**3.** Prove that given 13 points in the plane with integer coordinates, one can always find 4 of them such that their arithmetic mean has integer coordinates. (The arithmetic mean of points $a_1, \ldots, a_n$ is the point $\frac{1}{n}(a_1 + \cdots + a_n)$.)

**4.** In a convex polygon in the plane, there are at least $m^2+1$ points with integer coordinates. Prove that there are at least $m+1$ points with integer coordinates lying on the same line.

**5.** Several football teams are playing in a round-robin tournament (meaning that every pair of teams will eventually play one game). Prove that at any point during the tournament, there will be two teams that have played the same number of games.

**6.** Show that for any set of $n+1$ integers, there is always a non-empty subset whose sum is divisible by $n$.
**Hint.** Among any $n+1$ integers, there are two whose difference is divisible by $n$.

**7.** Show that a among any $n+2$ integers either there are two whose sum is divisible by $2n$ or there are two whose difference is divisible by $2n$.

**8.** For any integers $n$ and $u$, there exist integers $x$ and $y$, not both zero, such that:

$$-\sqrt{n} \leq x \leq \sqrt{n}, \quad -\sqrt{n} \leq y \leq \sqrt{n}$$

and $x - uy$ is divisible by $n$.

**9.** A sequence of real numbers $a_1, a_2, a_3, \ldots, a_{2024}$ is given. Prove that it is possible to select one or more consecutive numbers such that their sum differs from an integer by less than 0.001.

**10.** Show that if we are given 50 segments in a line, then there are 8 of them which are pairwise disjoint or 8 of them with a common point.

**11.** (non-trivial) Show that in any sequence of $ab+1$ different numbers there is either a subsequence of length $a+1$ which is increasing or a subsequence of length $b+1$ which is decreasing